
Fictitious Fluid Approach and Anomalous Blow-up of the Dissipation Rate in a 2D Model of Concentrated Suspensions

Leonid Berlyand¹, Yuliya Gorb², and Alexei Novikov²

¹ Department of Mathematics & Materials Research Institute, Pennsylvania State University, McAllister Bld., University Park, PA 16802, USA, berlyand@math.psu.edu

² Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA, gorb@math.psu.edu, anovikov@math.psu.edu

Summary. We present a two-dimensional (2D) mathematical model of a highly concentrated suspension or a thin film of the rigid inclusions in an incompressible Newtonian fluid. Our objectives are two-fold: (i) to obtain all singular terms in the asymptotics of the overall viscous dissipation rate as the interparticle distance parameter δ tends to zero, (ii) to obtain a qualitative description of a microflow between neighboring inclusions in the suspension.

Due to reduced analytical and computational complexity, 2D models are often used for a description of 3D suspensions. Our analysis provides the limits of validity of 2D models for 3D problems and highlights novel features of 2D physical problems (e.g. thin films). It also shows that the Poiseuille type microflow contributes to a singular behavior of the dissipation rate. We present examples in which this flow results in anomalous rate of blow up of the dissipation rate in 2D. We show that this anomalous blow up has no analog in 3D.

While previously developed techniques allowed to derive and justify the leading singular term only for special symmetric boundary conditions, a *fictitious fluid approach*, developed in this paper, captures *all* singular terms in the asymptotics of the dissipation rate for generic boundary conditions. This approach seems to be an appropriate tool for rigorous analysis of 3D models of suspensions as well as various other models of highly packed composites.

Key words: concentrated suspensions, overall or effective viscous dissipation rate, Stokes flow, discrete network approximation, variational bounds, Poiseuille flow.

1 Introduction

Many classical and novel engineering processes involving multiphase flow require to capture the overall behavior of suspensions. The problem of the be-

havior of suspensions is important in geophysics (mud-flow and debris flow rheology), pharmacology (drugs design), ceramics processing among others. Wide range of experimental (see, e.g., [1, 3, 4, 20, 31, 37, 44–46, 52]), and numerical (see, e.g., [17, 35, 40, 49, 53, 55]) results are available.

Two-dimensional (2D) models of three-dimensional (3D) suspensions are often used in numerical simulations (e.g. [21, 34, 47]), because they require less computational effort. In addition, 2D suspensions can describe biological thin films [56] and certain types of 3D concentrated suspensions of uniaxial thin rods (see e.g. [39]).

We consider a 2D mathematical model of a non-colloidal (Browning motion can be neglected) concentrated suspension of neutrally buoyant rigid particles (inclusions) in a Newtonian fluid. The suspension occupies a 2D domain Ω , and rigid inclusions are modeled by disks of equal radii. The main objective is to characterize in a rigorous mathematical framework the dependence of the effective rheological properties (e.g. effective viscosity, effective permeability, effective viscous dissipation rate) of such suspension on the geometry of inclusions array and applied boundary conditions on $\partial\Omega$. We focus on highly packed suspensions when the concentration of inclusions is close to maximal, which means that the distance between neighboring inclusions (*interparticle distance*) is much smaller than their sizes. We consider an irregular (non-periodic) array of disks and our analysis takes into account the variable distances between adjacent inclusions.

The main features of the problem under consideration are the high concentration of the inclusions and the irregular geometry of their spatial distribution. The key quantity of interest in describing the effective rheological properties of suspensions is the rate of viscous dissipation of energy $\widehat{W} = \widehat{W}(\mathbf{u})$, where \mathbf{u} is the velocity of the incompressible fluid (see the precise definition in Section 2).

Initially our interest was motivated by the problem of dewatering process which led to the sedimentation due to gravity problem [19] arising in the disposal of solid waste in industrial and municipal pollution management. In this problem particles are close to touching. In particular, shear external boundary condition was brought up by experimental and numerical studies [27, 29, 30] and it was addressed mathematically by analysis of 2D model in [10]. This led us to the question of limits of validity of 2D models for description of 3D suspension.

Note that the problem of sedimentation is quite difficult in mathematical content and far from being settled. Even in the case of the dilute limit this problem is not completely understood mathematically and we mention here the number of recent mathematical studies in this direction [7, 51].

For highly concentrated suspensions of rigid inclusions, \widehat{W} exhibits a singular behavior (see e.g. [8, 10, 24, 28, 50, 53]) and its understanding is a fundamental issue. A formal asymptotic analysis of the singular behavior of the viscous dissipation rate in a thin gap between a single pair of two closely spaced spherical inclusions in a Newtonian fluid was performed in [24]. In this

work only translational motions of inclusions but not rotational were considered and the asymptotics of the form $C\delta^{-1} + O(\ln 1/\delta)$, where δ is the distance between two spheres, was obtained. Based on the analysis of a single pair of spheres the authors of [24] suggested that the asymptotics of the effective viscosity of the periodic array of inclusions is of the same form, that is, the main singular behavior is of $O(\delta^{-1})$. Next, a periodic array of inclusions in a Newtonian fluid was considered in [50]. Under the assumption that all inclusions follow the shear motion of the fluid (formula (5) in [50]) it was shown that $\widehat{W} = O(\delta^{-1})$. This assumption is analogous to the well-known Cauchy-Born hypothesis in solid state physics, which is known to be *not* always true [25]. Indeed, in the case of suspensions it was shown in [10] that for shear external boundary conditions the inclusions may not follow the shear motion. Moreover, it was shown in numerical studies of [53] that the asymptotics $O(\delta^{-1})$ may or may not hold for suspensions of a large number of inclusions with generic boundary conditions. There it was observed numerically that while in some cases the asymptotics of the effective viscosity is of order $1/\delta$, in other cases it is of order $\ln 1/\delta$. Also the problem of the exact analytical form of the singular behavior for generic suspensions was posed in [53] (p. 140) which motivated subsequent studies of [8, 10] and the present paper, where a network approximation approach was developed for this type of problems.

In [8] a formal asymptotic analysis of the effective viscosity in 3D for a disordered array of inclusions was performed. In a view of discrepancies between predictions of the formal asymptotic analysis [24] and numerics [53] such formal asymptotics requires a mathematical justification. In [8] for special (extensional) boundary conditions the leading term of the asymptotics of the effective viscosity as $\delta \rightarrow 0$, where δ is the characteristic spacing between neighboring inclusions, was justified in a 2D model. In subsequent work [10] this leading term, that exhibits a so-called *strong blow up* of order $\delta^{-3/2}$, was analyzed. It turned out that in many important cases, e.g. when shear boundary conditions are applied, this term degenerates, so the next term of order $\delta^{-1/2}$, that exhibits a so-called *weak blow up*, becomes the leading term of the asymptotics in many physical situations.

However, the techniques of [8] are only capable to capture the strong blow up but not the other singular terms in the asymptotic expansion of the effective viscosity. By contrast, a *fictitious fluid approach* introduced in this paper allows to derive the complete asymptotic expansion of the overall viscous dissipation rate in which all singular terms are captured and justified. In particular, we ruled out singular terms other than presented in Theorem 2.1. Previously [8–10, 24, 28, 50, 53] inclusions in a dense suspension were characterized by the sets of their translational and rotational velocities, so-called discrete variables. Our analysis shows that in order to obtain the complete asymptotics of singular behavior it is necessary use an additional set of discrete variables, permeation constants. To the best of our knowledge these discrete variables have not been used in previous studies of dense suspensions. We now explain the physical consequences of this asymptotics.

The key feature of rheology of concentrated suspensions is that the dominant contribution to the overall or effective viscous dissipation rate (and therefore, to the effective viscosity) comes from thin gaps (lubrication regions) between closely spaced neighboring inclusions [43]. The mathematical techniques introduced in the above mentioned works [8, 24, 50] utilized this observation. More specifically, they took into account certain types of relative movements of inclusions which resulted in the corresponding microflows in the gaps. The formal asymptotics in [24] was based on analysis of the squeeze motion, when two inclusions move toward each other along the line joining their centers (see Fig. 2.7c) but did not provide the detailed analysis of other motions, which was sufficient for certain type of boundary conditions (e.g. extensional boundary conditions) but not sufficient for others. A justification of the formal asymptotics was not considered in [24] (see also [28] where similar results were obtained).

In [8] four types of relative motions of neighboring inclusions were considered which result in a singular behavior of dissipation rate: the squeeze (Fig. 2.7c), the shear (Fig. 2.7b) and two rotations (Fig. 2.8). While it was sufficient for the analysis of the leading term of the overall viscous dissipation rate (in a suspension of free particles) which was the goal of [8], in the present paper we observed that this analysis does not provide a complete picture of microflows. Indeed, the Poiseuille flow in 2D also contributes into the singular behavior. Examples in the present paper suggest that when an external field is applied to inclusions in a suspension, this Poiseuille flow may give rise to an anomalous strong rate of blow up (called a *superstrong blow up*, of order $\delta^{-5/2}$) of the viscous dissipation rate, whereas for suspensions of free inclusions there is at most strong blow up (of order $\delta^{-3/2}$).

As explained above the previous studies [8, 24, 28, 50] provided only a partial analysis of the single behavior of the viscous dissipation rate and therefore did not provide the complete physical picture of microflows. In this work we obtain the complete asymptotic description of the singular behavior of viscous dissipation rate which led us to the complete description of microflows (Fig. 2.7-2.10).

The techniques of [8, 10] and the present paper are based on the discrete network approximation. Discrete networks have been used as analogues of the continuum problems in various areas of physics and engineering for a long time (see, for example, [12] and references therein, and see also the recent review [48] for various applications of networks in social and biological studies). However, the fundamental issue of relationship between a continuum problem and the corresponding discrete network was not addressed until recently.

The pioneering study of electro-magnetic properties of high contrast materials arising in imaging using a rigorous network approximation was done in [14–16]. There the local resistivity of a periodic medium with continuously distributed properties was given by the Kozlov’s function ([38]): $\rho(\mathbf{x}) = \text{Const } e^{-S(\mathbf{x})/\varepsilon^2}$, where $S(\mathbf{x})$ is a periodic smooth phase function, ε is an aspect ratio of the material properties, a small parameter of the problem

(as $\varepsilon \rightarrow 0$, $\rho(\mathbf{x})$ describes a resistivity of a high contrast composite). The effective resistivity obtained and justified there was given in terms of the principal curvatures κ^+ , κ^- of the function $S(\mathbf{x})$ at the saddle point of $S(\mathbf{x})$.

The discrete network approximation for a medium with piecewise constant characteristics, which correspond to particle-filled (particulate) composites, was developed in [12]. In this case, the principle curvatures and saddle points of the phase function are not determined. In [12] an infinite contrast material ($\varepsilon = 0$) with irregularly or non-periodically distributed circular fibers was considered and asymptotics of the effective conductivity of such a composite as $\delta \rightarrow 0$ was derived and justified. In [12] the Keller's asymptotic solution [36] for two closely spaced disks was used as a building block for construction of the network for a large number of inclusions (disks). The network model is a linear system on a graph with appropriate boundary conditions. The sites or vertices of the graph correspond to the centers of inclusions $\{\mathbf{x}_i, i = 1, \dots, N\}$ and the values of the local fluxes between adjacent inclusions \mathbf{x}_i and \mathbf{x}_j described by the Keller's formula: $\pi \sqrt{\frac{R}{\delta_{ij}}}$, where R is the radius of disks, δ_{ij} is the distance between the closely spaced neighbors, are assigned to the edges of the graph. The notion of the neighbors is introduced via the Voronoi tessellation. The asymptotic closeness of the effective conductivity of continuum and discrete problems was justified by using the direct and dual variational bounds matching to the leading order which was obtained to be of order $\delta^{-1/2}$ as $\delta \rightarrow 0$, where $\delta = \max \delta_{ij}$.

In [13] the notion of the characteristic interparticle distance was generalized for a broad class of 2D geometrical patterns (highly non-uniform arrays of inclusions). For this class of arrays an explicit error estimate for the discrete network approximation was obtained. A scalar conductivity problem analogous to [12, 13] in 3D was considered in [11] where both asymptotics of the effective conductivity and its error estimated were obtained. While the formulation of the problem is analogous to the 2D one considered in [12, 13], the connectivity patterns of conducting inclusions are much more complex in 3D. That is why the justification of this approximation and derivation of its error estimate required introduction of new techniques (e.g., the central projection partition). By contrast, our work shows that 2D vectorial problems may exhibit features which have no analogs in similar 3D problems.

Next the method of the discrete network approximation was extended to a vectorial problem [8], where formal asymptotics for 3D suspensions was derived. The approach developed in [12, 13] for the justification of the formal asymptotics in the scalar problem could not be readily applied to the vectorial problems, which exhibit new features and require new technical tools. In particular, the key point of discrete network approach is a construction of trial functions in the variational formulation of the problem for determining the effective properties of composites. In vectorial problems (unlike in scalar ones) such trial functions (for the direct variational principle) must satisfy the divergence free condition. Also the dual variational formulation involves

integral constraints - balances of linear and angular momenta. The balance equations on an inclusion involve all its neighbors that have their own neighbors etc. This accounts for a long-range hydrodynamic interactions and shows that the pairwise (local) analysis (such as [24, 28]) is not sufficient and a global analysis is necessary for the whole array of inclusions. That is why a generalization of scalar techniques from [12, 13] was done for the leading term only in the asymptotics of the effective viscosity in the 2D model [8]. It seems that a straightforward generalization of the approach in [8] becomes increasingly difficult and probably impossible due to the global nature of above mentioned constraints.

The objectives of this work are two-fold: (i) develop a new approach, called a *fictitious fluid approach*, that allows to effectively deal with global constraints (balance equations and the divergence free condition), (ii) apply the fictitious fluid approach to derive and justify *all* singular terms in the asymptotics of the overall viscous dissipation rate. Such an asymptotic formula results in a *complete* description of microflows in suspensions, as oppose to the partial description of the microflow in [8, 24, 28, 50].

The techniques in [11–13] for scalar problems in both 2D and 3D were based on a direct (one-step) discretization of the original continuum problem. In vectorial problems [8] the direct discretization allowed to obtain only the leading term for the extensional viscosity (strong blow up) but not for the shear viscosity (weak blow up) due to the global constraints. In the present work we use a two-step discretization procedure based on the fictitious fluid approach to derive and justify both strong and weak blow up terms.

We briefly describe now the idea behind the fictitious fluid approach. As mentioned before, it consists of two steps. In step 1 we introduce a fictitious fluid domain which comprises thin gaps between neighboring inclusions. The dissipation rate restricted to this domain is denoted by $\widehat{W}_{\mathbf{\Pi}}$. We show that for generic Dirichlet boundary conditions it describes the singular behavior in the following sense:

$$\widehat{W} = \widehat{W}_{\mathbf{\Pi}} + h.o.t. \quad \text{as } \delta \rightarrow 0.$$

In step 2 we perform a discretization of $\widehat{W}_{\mathbf{\Pi}}$, that is, the continuum problem for $\widehat{W}_{\mathbf{\Pi}}$ is reduced to an algebraic problem on a graph, called a *network problem*. By using the fictitious fluid approach in this step most of the difficulties are eliminated. The network problem is a minimization of a quadratic form whose coefficients depend on R , μ and boundary data. The quadratic form on the minimizing set of discrete variables is called the *discrete viscous dissipation rate* and denoted by \mathcal{I} . Unknowns of this problem are vectors $\mathbb{U} = \{\mathbf{U}^i\}_{i=1}^N$, $\boldsymbol{\omega} = \{\boldsymbol{\omega}^i\}_{i=1}^N$, the translational and angular velocities of inclusions, respectively, and a collection of numbers $\boldsymbol{\beta} = \{\beta_{ij}\}$ characterizing the Poiseuille microflow between a pair of inclusions. The discrete dissipation rate \mathcal{I} is given by:

$$\mathcal{I} = \mathcal{I}_1(\boldsymbol{\beta})\delta^{-5/2} + \mathcal{I}_2(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta})\delta^{-3/2} + \mathcal{I}_3(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta})\delta^{-1/2}, \quad \text{as } \delta \rightarrow 0 \quad (1.1)$$

where \mathcal{I}_k , $k = 1, 2, 3$, are explicitly computable quadratic polynomials of \mathbb{U} , ω , β .

As a result we obtain the following asymptotic formula for the generic Dirichlet boundary conditions:

$$\frac{|\widehat{W} - \mathcal{I}|}{\mathcal{I}} = O(\delta^{1/2}) \quad \text{as } \delta \rightarrow 0.$$

In fact, we prove the following result about the error term:

$$|\widehat{W} - \mathcal{I}| \leq \mu \left(\sum_{i,j} \mathbb{E}_1(\beta_{ij}) + \mathbb{E}_2(\mathbf{U}^i - \mathbf{U}^j) + \mathbb{E}_3(\omega^i) \right)$$

where $\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3$ are quadratic polynomials of β_{ij} , difference $\mathbf{U}^i - \mathbf{U}^j$ and ω^i , respectively, whose coefficients are independent of δ .

Finally, we analyze the physical ramification of the obtained asymptotic formula by presenting several examples. We construct an example of a suspension in a strong “pinning” external field, where \widehat{W} is of order $\delta^{-5/2}$ (super-strong blow up). We also show an example of the superstrong blow up due to the boundary layer effect. Note that to the best of our knowledge this rate of blow up was not observed before and we call it an *anomalous rate*. For generic suspensions (free particles or a weak external field) we expect that $\beta = O(\delta)$ and, therefore, the first and the third terms of (1.1) are of the same order $O(\delta^{-1/2})$. For a hexagonal array of inclusions we prove that \widehat{W} exhibits the strong blow up (of order $\delta^{-3/2}$) and $\beta = \mathbf{0}$. Note that a typical close packing array in 2D is “approximately” hexagonal.

The paper is organized as follows. In Section 2 we give a mathematical formulation of the problem (Subsection 1), describe the fictitious fluid approach, and present our main results in Theorems 2.1, 2.2 (Subsection 2). In Subsection 3 we construct our discrete network and discuss how local flows in thin gaps between neighbors (microflows) contribute the overall viscous dissipation rate and state the theorem about a representation of the error term of the discrete approximation. In Section 3 we discuss main and present examples. Section 4 is devoted to the fictitious fluid problem. In Section 5 we present results related to our discrete network. In section 6 prove two lemmas about approximation of the overall dissipation rate by the dissipation rate of the fictitious fluid and about approximation of the fictitious fluid (continuum) dissipation rate by the discrete one. Conclusions are presented in Section 7. The proofs of some auxiliary facts are given in Appendices.

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2 Formulation of the Problem and Main Results

1 Mathematical Formulation of the Problem

Consider an irregular or non-periodic array of N identical circular disks B^i , of the radius R distributed in a rectangular domain Ω . Denote by $\Omega_F = \Omega \setminus \bigcup_{i=1}^N B^i$ the *fluid domain* which is occupied by incompressible fluid with viscosity μ (see Fig. 2.1). Disks B^i represent absolutely rigid inclusions. Inertia of both fluid and inclusions is neglected. In the fluid domain Ω_F consider the following boundary value problem:

$$\left\{ \begin{array}{ll} (a) \mu \Delta \mathbf{u} = \nabla p, & \mathbf{x} \in \Omega_F \\ (b) \nabla \cdot \mathbf{u} = 0, & \mathbf{x} \in \Omega_F \\ (c) \mathbf{u} = \mathbf{U}^i + R\omega^i(n_1^i \mathbf{e}_2 - n_2^i \mathbf{e}_1), \mathbf{x} \in \partial B^i, & i = 1 \dots N \\ (d) \int_{\partial B^i} \boldsymbol{\sigma}(\mathbf{u}) \mathbf{n}^i ds = \mathbf{0} & i = 1 \dots N \\ (e) \int_{\partial B^i} \mathbf{n}^i \times \boldsymbol{\sigma}(\mathbf{u}) \mathbf{n}^i ds = 0, & i = 1 \dots N \\ (f) \mathbf{u} = \mathbf{f}, & \mathbf{x} \in \partial\Omega \end{array} \right. \quad (2.1)$$

where $\mathbf{u}(\mathbf{x})$ is the velocity field at a point $\mathbf{x} \in \Omega_F$, $p(\mathbf{x})$ is the pres-

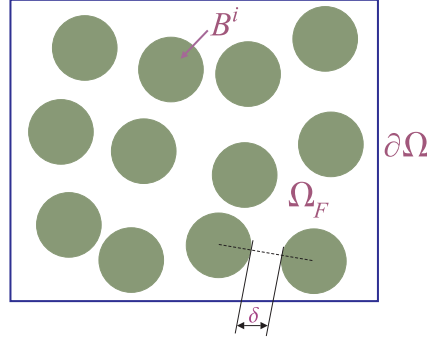


Fig. 2.1. Domain Ω_F occupied by the fluid of viscosity μ , and disordered array of closely spaced inclusions B^i

sure, $\boldsymbol{\sigma}(\mathbf{u}) = 2\mu\mathbf{D}(\mathbf{u}) - p\mathbf{I}$ is the stress tensor, $D_{ij}(\mathbf{u}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$,

$i, j = 1, 2$, is the rate of strain which satisfies the incompressibility condition: $\text{tr } \mathbf{D}(\mathbf{u}) = 0$, another form of (2.1b). The vector $\mathbf{n}^i = (n_1^i, n_2^i)$ is the outer normal to B^i . Constant vectors $\mathbf{U}^i = (U_1^i, U_2^i)$ and scalars ω^i , $i = 1, \dots, N$, which are translational and angular velocities of the inclusion B^i , respectively, are to be found in the course of solving the problem.

Here N is closed to maximal packing number $N_{max} = N_{max}(\Omega, R)$. This number is finite and $|N - N_{max}|$ depends on the small parameter δ called interparticle distance rigorously defined below in Definition 1.

We consider the linear external boundary conditions of the form:

$$\mathbf{f} = A\mathbf{x} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (2.2)$$

where the components a, b, c of the matrix A are given constants. Note that the most general form of the linear boundary conditions is $\mathbf{f} = \mathbf{f}_0 + A\mathbf{x}$ where \mathbf{f}_0 is a constant vector. Observe that when $a = 0$ and $b = -1/c$ the vector $A\mathbf{x}$ corresponds to a rotation and \mathbf{f}_0 to a translation of the boundary, hence, \mathbf{f} describes the rigid body motion of $\partial\Omega$. Hereafter, we exclude this trivial motion from our consideration assuming that $a = 0$ and $b = -1/c$ does not hold simultaneously in (2.2) and $\mathbf{f}_0 = \mathbf{0}$. We use such boundary conditions for two reasons: *a*) for technical simplicity, which does not lead to the loss of generality; *b*) they include the *shear* (when $a = c = 0$, $b = 1$) and *extensional* ($a = 1$, $b = c = 0$) boundary conditions (see e.g. [10]) which model two basic types of viscometric measurements (see Fig. 2.2). It is possible to extend our results to arbitrary Dirichlet boundary conditions $\mathbf{f} \in H^{1/2}(\partial\Omega)$ satisfying $\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n} ds = 0$.

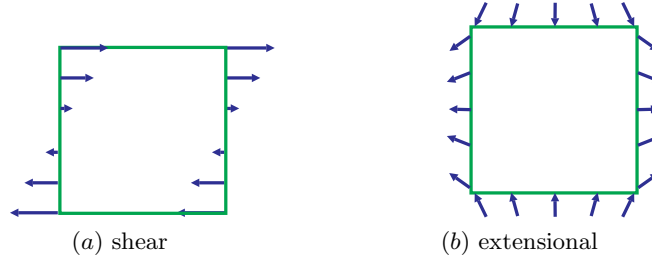


Fig. 2.2. Shear and extensional external boundary conditions

For an arbitrary set $A \subseteq \Omega_F$ consider the following integral:

$$\begin{aligned} W_A(\mathbf{v}) &= \frac{1}{2} \int_A \boldsymbol{\sigma}(\mathbf{v}) : \mathbf{D}(\mathbf{v}) = \mu \int_A \mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{v}) d\mathbf{x} \\ &= \mu \int_A \left[\left(\frac{\partial v_1}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x} \right)^2 + \left(\frac{\partial v_2}{\partial y} \right)^2 \right] d\mathbf{x} \end{aligned} \quad (2.3)$$

where $\mathbf{v} = (v_1, v_2)$. Then the variational formulation of (2.1) is:

$$\text{Find } \mathbf{u} \in V, \text{ such that } W_{\Omega_F}(\mathbf{u}) = \min_{\mathbf{v} \in V} W_{\Omega_F}(\mathbf{v}), \quad (2.4)$$

where the set of admissible vector fields V is defined by

$$\begin{aligned} V = \{ \mathbf{v} \in \mathbf{H}^1(\Omega_F) : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega_F, \mathbf{v} = \mathbf{f} \text{ on } \partial\Omega, \\ \mathbf{v} = \mathbf{U}^i + \boldsymbol{\omega}^i \times (\mathbf{x} - \mathbf{x}^i), \mathbf{x} \in \partial B^i, \quad i = 1, \dots, N \}, \end{aligned} \quad (2.5)$$

$W_{\Omega_F}(\mathbf{u})$ is called the (continuum) viscous dissipation rate [42] and it is the principal quantity of interest in the study of overall properties of suspensions. We will use the following notation:

$$\widehat{W} := W_{\Omega_F}(\mathbf{u}). \quad (2.6)$$

The key feature of our problem is that we study suspensions where concentration of inclusions is close to its maximum. Therefore, the domain Ω_F depends on the characteristic interparticle distance parameter δ . Our main objective is to derive and justify an asymptotics of \widehat{W} as $\delta \rightarrow 0$. We will show that the coefficients of this asymptotic formula are determined by the solution to a discrete network problem, which determine the discrete viscous dissipation rate. As explained in Introduction, \widehat{W} determines a number of important measurable properties of suspensions.

2 The Fictitious Fluid Approach and Discretization

In Appendix 1 we show that using the notion of Voronoi tessellation we can decompose the domain Ω_F into *necks* \mathbf{II} and *triangles* $\mathbf{\Delta}$: $\Omega_F = \mathbf{II} \cup \mathbf{\Delta}$ (see Fig. 2.4). Necks connect either two disks (Fig. 2.3a) or a disk and a part of the boundary $\partial\Omega$ called a *quasidisk* (see Fig. 2.3b), that is, necks connect *neighbors*. The velocities of quasidisks are given by the prescribed boundary conditions (2.2). Note that near the boundary when quasidisks are involved the “triangles” are actually trapezoids. With slight abuse of terminology, we also call them *triangles*.

We distinguish boundary disks (quasidisks) and interior disks and introduce two sets of the corresponding indices. For indices of interior disks we use the notation $\mathbb{I} = \{1, \dots, N\}$. If the disk B^i centered at \mathbf{x}_i is a quasidisk then i belongs to the set \mathbb{B} of the indices of quasidisks. Also denote by \mathcal{N}_i the set of indices of all neighbors of B^i .

For a given array of the disks and quasidisks B^i centered at \mathbf{x}_i , the **discrete network** is the graph $\mathcal{G} = (\mathcal{X}, \mathcal{E})$, with set of vertices $\mathcal{X} = \{\mathbf{x}_i : i \in \mathbb{I} \cup \mathbb{B}\}$ and set of edges $\mathcal{E} = \{e_{ij} : i \in \mathbb{I}, j \in \mathcal{N}_i\}$ with e_{ij} connecting neighbors B^i and B^j .

As mentioned in Introduction our main approach in study of the asymptotics of \widehat{W} as $\delta \rightarrow 0$ consists of two steps. This two-step approach allows to

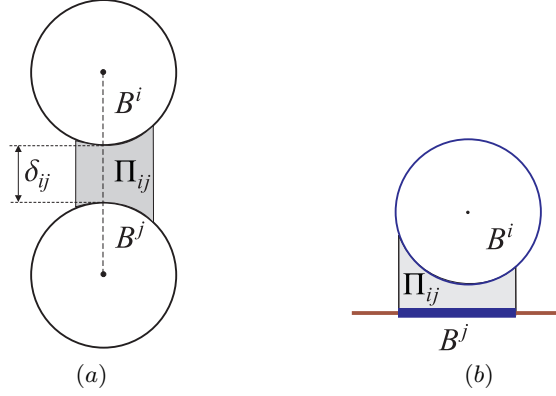


Fig. 2.3. (a) neck connecting two disks; (b) neck connecting disk B^i and quasidisk B^j

separate geometric construction of the network and its subsequent asymptotic analysis.

In the first step, we show that the minimization problem (2.4) in the fluid domain Ω_F can be approximated by a “fictitious fluid” problem in which fluid is assumed to occupy necks $\mathbf{\Pi} = \bigcup_{i \in \mathbb{I}, j \in \mathcal{N}_i} \Pi_{ij}$ between closely spaced neighbor-

ing inclusions (the shadowed region in Fig. 2.4). We call $\mathbf{\Pi}$ the *fictitious fluid domain*. On the boundary of the complementary part of the domain (triangles in Fig. 2.4) the relaxed incompressibility conditions:

$$\int_{\partial \Delta_{ijk}} \mathbf{v} \cdot \mathbf{n} ds = 0, \quad i \in \mathbb{I}, \quad j, k \in \mathcal{N}_i, \quad (2.7)$$

are imposed. This reflects a well-known physical fact that for densely packed suspensions the dominant contribution to the viscous dissipation rate over the fluid domain comes from those necks.

Below we show that the functional \widehat{W} is decomposed as follows:

$$\widehat{W} = \widehat{W}_{\mathbf{\Pi}} + \widehat{W}_{\Delta}, \quad (2.8)$$

where $\widehat{W}_{\mathbf{\Pi}}$ is the overall viscous dissipation rate of the fictitious fluid defined below in (2.10) and \widehat{W}_{Δ} is the remaining contribution from the domain Δ .

Consider the problem of minimization of the functional $W_{\mathbf{\Pi}}$, defined by (2.3) over the fictitious fluid domain $\mathbf{\Pi}$, in the following class of functions:

$$\begin{aligned} V_{\mathbf{\Pi}} = \{ \mathbf{v} \in \mathbf{H}^1(\mathbf{\Pi}) : \nabla \cdot \mathbf{v} = 0 \text{ in } \mathbf{\Pi}, \int_{\partial \Delta_{ijk}} \mathbf{v} \cdot \mathbf{n} ds = 0 \text{ for all } \Delta_{ijk} \in \Delta, \\ \mathbf{v} = \mathbf{U}^i + R\omega^i(n_1^i \mathbf{e}_2 - n_2^i \mathbf{e}_1), \mathbf{x} \in \partial B^i, i = 1, \dots, N, \mathbf{v} = \mathbf{f} \text{ on } \partial \mathbf{\Pi} \cap \partial \Omega \}, \end{aligned} \quad (2.9)$$

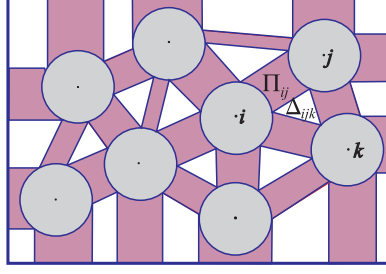


Fig. 2.4. The decomposition of the original fluid domain Ω_F into the fictitious fluid domain and the set of triangles

where \mathbf{U}^i and ω^i , $i = 1, \dots, N$, are arbitrary constant vectors and arbitrary constants, respectively. The zero-flux condition through $\partial\Delta_{ijk}$ (2.7) is inherited from the problem in the original fluid domain Ω_F due to incompressibility condition of the fluid in triangles. Such a condition is a necessary (but not sufficient) for $\nabla \cdot \mathbf{v} = 0$ in the triangle Δ_{ijk} .

We define the overall dissipation rate of the fictitious fluid by

$$\widehat{W}_{\Pi} = \min_{\mathbf{v} \in V_{\Pi}} W_{\Pi}(\mathbf{v}). \quad (2.10)$$

The first principal result of this paper is that the dissipation rate \widehat{W} can be approximated by the rate \widehat{W}_{Π} and \widehat{W}_{Δ} can be neglected. To show this we need to introduce a small parameter of the problem, which is a characteristic interparticle distance δ .

For each pair of neighbors $\mathbf{x}_i, \mathbf{x}_j$ define

$$\delta_{ij} = \begin{cases} |\mathbf{x}_i - \mathbf{x}_j| - 2R, & \text{when } i, j \in \mathbb{I}, \\ |\mathbf{x}_i - \mathbf{x}_j| - R, & \text{when either } i \text{ or } j \in \mathbb{B}, \\ |\mathbf{x}_i - \mathbf{x}_j|, & \text{when } i, j \in \mathbb{B}. \end{cases} \quad (2.11)$$

As mentioned above, we study domains with closely spaced neighboring disks. More precisely, for all pairs of neighbors we assume that following *close-packing condition* holds.

Definition 1. Write the minimal distance δ_{ij} (see Fig. 2.3) between any two neighboring disks B^i and B^j in the form $\delta_{ij} = \delta d_{ij}$, where d_{ij} is such that $0 < c_1 < d_{ij} < c_2$ for some absolute constants c_1, c_2 . If the parameter δ , called the **characteristic interparticle distance**, tends to zero, then Ω_F is said to satisfy the **close-packing condition**.

We remark that this definition describes uniformly dense arrays of disks. A more general definition which covers a notion of a “hole” corresponding to the void space in the composite is introduced and discussed e.g. in [11, 13].

Hereafter we call the array of inclusions under consideration a *quasi-hexagonal array* (e.g. in [13] it is referred to as “randomized hexagonal”). Recall, that for such arrays all neighbors are closely spaced and a typical number of nearest neighbors for a disk is six.

Hence, the mathematical thrust of step one of the fictitious fluid approach is in showing that the overall rate of the energy dissipation \widehat{W}_{Π} of the fictitious fluid captures the singular behavior of \widehat{W} , defined by (2.6), as $\delta \rightarrow 0$. More precisely, in Section 5 we prove the following theorem:

Theorem 2.1 (Approximation by the fictitious fluid). *Suppose an array of inclusions satisfies the close packing condition. Let \widehat{W} be the overall viscous dissipation rate defined by (2.5)-(2.6) and \widehat{W}_{Π} be the viscous dissipation rate of the fictitious fluid defined by (2.10). Then the following asymptotic formula holds:*

$$\frac{|\widehat{W} - \widehat{W}_{\Pi}|}{\widehat{W}_{\Pi}} = O(\delta^{1/2}) \quad \text{as } \delta \rightarrow 0. \quad (2.12)$$

In step two, we study asymptotics (blow up) of the overall viscous dissipation rate \widehat{W} as $\delta \rightarrow 0$. In view of step one, this is reduced to finding of asymptotics of \widehat{W}_{Π} . The latter is done by a construction of a discrete network approximation and introduction of a so-called discrete viscous dissipation rate \mathcal{I} . To show closeness of the continuum and the discrete dissipation rates, \widehat{W}_{Π} and \mathcal{I} , respectively, we employ the direct and dual variational techniques [8, 12, 13].

Also note that the conditions (2.1d,e) in the original problem led to significant technical difficulties in variational analysis of the overall viscous dissipation rate of [8], which is why the analysis of [8] is restricted to its leading singular term. In contrast, analogs of these conditions in the fictitious fluid problem are satisfied *automatically* by construction, which results in substantial simplification of the analysis and thus allows to capture all singular terms.

The approximation of the overall viscous dissipation rate \widehat{W} by the discrete dissipation rate \mathcal{I} is given by the following theorem.

Theorem 2.2 (Approximation of the continuum dissipation rates by the discrete one). *Suppose Ω_F satisfies the close packing condition and $\mathcal{I} = \min_{(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}) \in \mathcal{R}} Q(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta})$, where the positive definite quadratic form*

$$Q(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}) = \mathcal{I}_1(\boldsymbol{\beta})\delta^{-5/2} + \mathcal{I}_2(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta})\delta^{-3/2} + \mathcal{I}_3(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta})\delta^{-1/2}$$

on the class of admissible discrete variables $(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}) \in \mathcal{R}$ defined in (2.33)-(2.37). Then the following approximation to the viscous dissipation rate holds:

$$\frac{|\widehat{W} - \mathcal{I}|}{\mathcal{I}} = O(\delta^{1/2}) \quad \text{as } \delta \rightarrow 0, \quad (2.13)$$

Remark 2.1. \mathcal{I}_i ($i = 1, 2, 3$) are explicitly computable quadratic polynomials of $(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}) \in \mathcal{R}$ from equations (2.36)-(2.37) below. These polynomials depend only on boundary data \mathbf{f} , viscosity μ and geometry of Ω_F .

The next subsection is devoted to the construction of these quadratic polynomials. We first introduce a set of discrete variables $(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta})$, define a new variable β_{ij} in each neck Π_{ij} and explain how the quadratic form Q is obtained. We will also discuss the underlying structure of the flow in a neck (microflow) and physical ramifications of the obtained asymptotics (2.13) and (2.38) below.

3 Construction of the discrete network. Microflows.

(a) We begin with the discretization of the boundary conditions. Denote $\mathbb{U} = \{\mathbf{U}^i\}_{i \in \mathbb{I} \cup \mathbb{B}}$, $\boldsymbol{\omega} = \{\omega^i\}_{i \in \mathbb{I} \cup \mathbb{B}}$ where on the boundary $\partial\Omega$, that is, for $i \in \mathbb{B}$, these velocities are given by boundary conditions as follows:

$$\mathbf{U}^i = \begin{cases} \begin{pmatrix} ax \\ cx \end{pmatrix}, & \mathbf{x}_i \in \partial\Omega_{lat} \\ \begin{pmatrix} by \\ -ay \end{pmatrix}, & \mathbf{x}_i \in \partial\Omega^\pm \end{cases} \quad \text{and} \quad \omega^i = \begin{cases} c, & \mathbf{x}_i \in \partial\Omega^- \\ -c, & \mathbf{x}_i \in \partial\Omega^+ \\ b, & \mathbf{x}_i \in \partial\Omega_{lat}^- \\ -b, & \mathbf{x}_i \in \partial\Omega_{lat}^+ \end{cases} \quad (2.14)$$

where \mathbf{x}_i is the center of the quasidisk B^i and $\partial\Omega^+$ and $\partial\Omega^-$ are the upper and lower parts of $\partial\Omega$, respectively, and $\partial\Omega_{lat}^+$ and $\partial\Omega_{lat}^-$ are the lateral (left and right) boundary.

(b) Discretization of the incompressibility condition is implemented as follows. Decompose the domain Ω_F into curvilinear hexagons \mathcal{A}_{ijk} as in Fig. 2.5:

$$\Omega_F = \bigcup_{i \in \mathbb{I}, j, k \in \mathcal{N}_i} \mathcal{A}_{ijk}.$$

Each \mathcal{A}_{ijk} consists of the line segments ℓ_{ij} , ℓ_{jk} , ℓ_{ki} (Fig. 2.5a) connecting disks B^i , B^j , B^k and arcs a_i , a_j , a_k of the corresponding disk.³ Then the weak incompressibility condition (2.7) for the class $V_{\mathbf{H}}$ (2.9) becomes

$$\int_{\mathcal{A}_{ijk}} \mathbf{u} \cdot \mathbf{n} ds = 0, \quad \text{for any } \mathcal{A}_{ijk}, \quad i \in \mathbb{I}, j, k \in \mathcal{N}_i. \quad (2.15)$$

In order to continue our analysis at this point we must introduce a new set of discrete variables. Here we define permeation constants:

$$\begin{aligned} \beta_{ij}^* &= \frac{1}{R} \int_{\ell_{ij}} \mathbf{u} \cdot \mathbf{n} ds, \quad i \in \mathbb{I}, \quad j \in \mathcal{N}_i, \\ \beta_{ij}^* &= \frac{1}{R} \int_{\ell_{ij}} \mathbf{f} \cdot \mathbf{n} ds, \quad \ell_{ij} \subset \partial\Omega \quad (i, j \in \mathbb{B}), \end{aligned} \quad (2.16)$$

³ In the case when the disk B^i has two quasidisk neighbors B^j and B^k , that is, when B^k is in the corner as in Fig. 2.5b, then the domain \mathcal{A}_{ijk} is actually a curvilinear pentagon. By slight abuse of terminology we still call it a “curvilinear hexagon”.

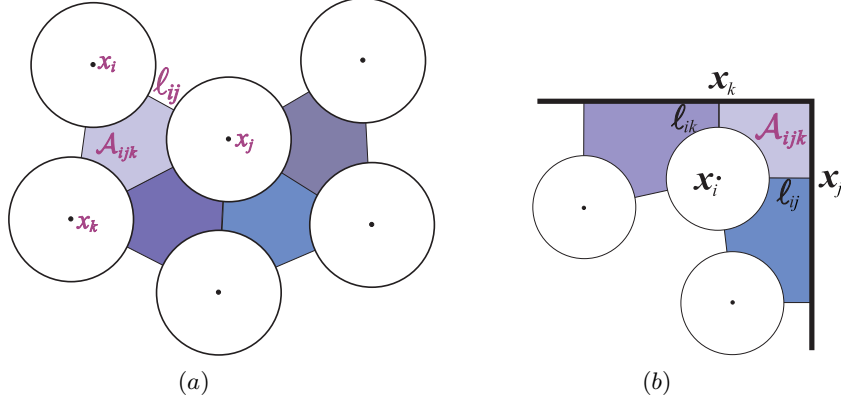


Fig. 2.5. (a) Decomposition of Ω_F into curvilinear hexagons \mathcal{A}_{ijk} and line ℓ_{ij} connecting neighbors B^i and B^j , (b) Construction of \mathcal{A}_{ijk} at the corner of the boundary $\partial\Omega$

where ℓ_{ij} is the line segment joining two neighbors B^i and B^j , $\mathbf{u} \in V_\Pi$ and \mathbf{n} is an outer normal to \mathcal{A}_{ijk} . Then (2.15) can be rewritten as:

$$\beta_{ij}^* + \beta_{jk}^* + \beta_{ki}^* + \frac{1}{R} \int_{a_i} \mathbf{U}^i \cdot \mathbf{n}^i ds + \frac{1}{R} \int_{a_j} \mathbf{U}^j \cdot \mathbf{n}^j ds + \frac{1}{R} \int_{a_k} \mathbf{U}^k \cdot \mathbf{n}^k ds = 0,$$

for $i \in \mathbb{I}$, $j, k \in \mathcal{N}_i$ (\mathbf{n}^i is a unit outer normal to ∂B^i), which can be further simplified as

$$\beta_{ij}^* + \beta_{jk}^* + \beta_{ki}^* + (\mathbf{U}^i + \mathbf{U}^j) \mathbf{p}^{ij} + (\mathbf{U}^j + \mathbf{U}^k) \mathbf{p}^{jk} + (\mathbf{U}^k + \mathbf{U}^i) \mathbf{p}^{ki} = 0, \quad (2.17)$$

where vectors \mathbf{q}^{ij} and \mathbf{p}^{ij} are the unit vectors of the local system coordinate of two neighboring disks B^i and B^j as in Fig. 2.6. We call (2.17) the *weak incompressibility condition*. This formula explains the scaling $\frac{1}{R}$ in (2.16). In-

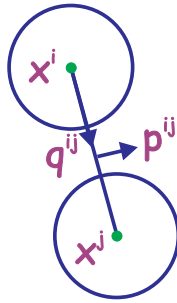


Fig. 2.6. Unit vectors of the local coordinate system

deed, from dimensional analysis $\int_{\ell_{ij}} \mathbf{u} \cdot \mathbf{n} ds$ must be divided by a characteristic lengthscale which is R in the curvilinear hexagonal \mathcal{A}_{ijk} .

(c) We finally discretize the Stokes equations in necks. This discretization is based on lubrication theory [2, 6, 43]. This theory describes thin film flows between two solid bodies sliding relative to each other. It is well-known (see e.g. [43]) that in dense suspensions the dominant hydrodynamic contribution to the viscous dissipation rate occurs in necks between closely spaced inclusions, where lubrication equations are relevant. Lubrication theory determines fluid motion in such necks as a result of relative kinematic motion of the neighboring inclusions. This raises a question of the classification of *microflows*, that is, local flows in necks between two closely spaced neighbors.

Recall three classical types of microflows between two parallel plates: the shear, the squeeze, and the Poiseuille flow. The last one, however, is not related to motions of two plates relative one to another and therefore it is not described by lubrication theory. In this work we show that exactly these three types of microflows fully describe the motion of the fluid between two neighboring disks. Asymptotic analysis of microflows between two parallel plates technically is much simpler than that for inclusions with curvilinear boundaries, which is needed for suspensions.

As shown below, in classical 3D problem the Poiseuille microflow between two inclusions does not contribute to the singular behavior of the viscous dissipation rate. However, in analogous 2D problem (e.g. thin films) all three microflows are present and, moreover, the Poiseuille flow results in anomalously strong singularity. Note that in the previous studies of the overall properties of suspensions the Poiseuille type microflow was not taken into account.

Thus, it is necessary to analyze kinematics of a pair of neighboring inclusions when one moves relative to the other. To this end for a pair of neighbors B^i and B^j , centered at \mathbf{x}_i and \mathbf{x}_j , we choose the *local* coordinate system where the origin is at $(\mathbf{x}_i + \mathbf{x}_j)/2$ and the y -axis is directed along the vector connecting \mathbf{x}_i and \mathbf{x}_j .

For clarity of presentation we consider two interior disks only (that is, $i \in \mathbb{I}$, $j \in \mathcal{N}_i \cap \mathbb{I}$). An analogous construction in boundary necks ($i \in \mathbb{I}$, $j \in \mathcal{N}_i \cap \mathbb{B}$) is given in Appendix 2.

There are exactly five elementary kinematic motions of inclusions. To see this we consider boundary conditions on ∂B^i , ∂B^j in (2.9) and first assume that $\omega^i = \omega^j = 0$. Then the conditions: $\mathbf{u} = \mathbf{U}^i$ on ∂B^i and $\mathbf{u} = \mathbf{U}^j$ on ∂B^j can be rewritten as follows:

$$\begin{aligned} \mathbf{u}|_{\partial B^i} &= \frac{1}{2}(\mathbf{U}^i + \mathbf{U}^j) + \frac{1}{2}(U_1^i - U_1^j)\mathbf{e}_1 + \frac{1}{2}(U_2^i - U_2^j)\mathbf{e}_2, \\ \text{and } \mathbf{u}|_{\partial B^j} &= \frac{1}{2}(\mathbf{U}^i + \mathbf{U}^j) - \frac{1}{2}(U_1^i - U_1^j)\mathbf{e}_1 - \frac{1}{2}(U_2^i - U_2^j)\mathbf{e}_2. \end{aligned}$$

Hence, the translational velocities of disks are decomposed into three motions. First, when both inclusions and fluid move with the same velocity (see Fig. 2.7a), and therefore this motion does not contribute to the singular behavior

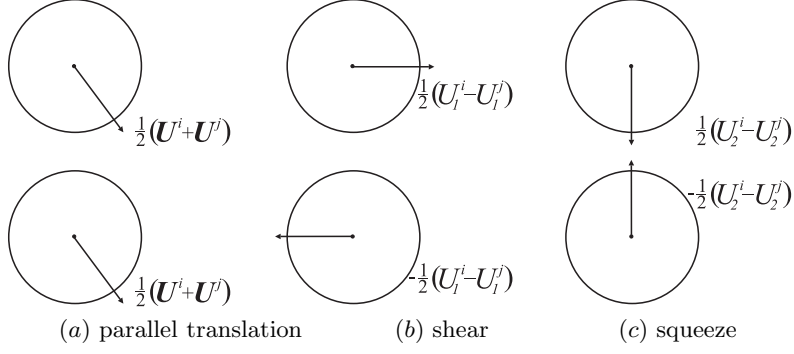


Fig. 2.7. Decomposition of translational velocities \mathbf{U}^i and \mathbf{U}^j into three elementary motions

of the viscous dissipation rate. Second, when one inclusion moves relative to the other producing the *shear* type motion (Fig. 2.7b) and, finally, the *squeeze* type motion (Fig. 2.7c) of the fluid.

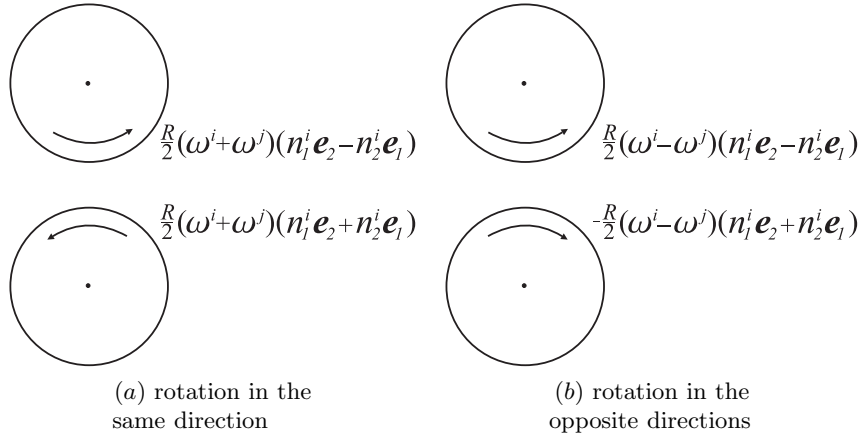


Fig. 2.8. Decomposition of the angular velocities ω^i and ω^j into two elementary motions

Similarly, assuming $\mathbf{U}^i = \mathbf{U}^j = \mathbf{0}$ in conditions on ∂B^i , ∂B^j in (2.9) we decompose them into two relative elementary motions: rotations of the disks in the same direction (Fig. 2.8a), and rotations of the disks in the opposite directions (Fig. 2.8b). Thus,

$$\mathbf{u}|_{\partial B^i} = \frac{R}{2}(\omega^i + \omega^j)(n_1^i \mathbf{e}_2 - n_2^i \mathbf{e}_1) + \frac{R}{2}(\omega^i - \omega^j)(n_1^i \mathbf{e}_2 - n_2^i \mathbf{e}_1)$$

and $\mathbf{u}|_{\partial B^j} = \frac{R}{2}(\omega^i + \omega^j)(n_1^i \mathbf{e}_2 + n_2^i \mathbf{e}_1) + \frac{R}{2}(\omega^i - \omega^j)(-n_1^i \mathbf{e}_2 - n_2^i \mathbf{e}_1),$

where $\mathbf{n}^i = (n_1^i, n_2^i)$ is the outer normal to ∂B^i .

Next consider microflows corresponding to the four kinematic motions contributing to the singular behavior of the dissipation rate described above. We further choose the microflow \mathbf{u}_{lub} which minimizes the viscous dissipation rate in the neck Π_{ij} by imposing the natural boundary conditions on the lateral boundaries $\partial \Pi_{ij}^\pm$ (Fig. 2.9). That is, the function \mathbf{u}_{lub} minimizes $W_{\Pi_{ij}}$ in the class:

$$\mathcal{V}_{ij} = \{ \mathbf{v} \in H^1(\Pi_{ij}) : \nabla \cdot \mathbf{v} = 0 \text{ in } \Pi_{ij}, \mathbf{v} = \mathbf{g}_i \text{ on } \partial B^i, \mathbf{v} = \mathbf{g}_j \text{ on } \partial B^j \} \quad (2.18)$$

$$\text{with } \mathbf{g}_i = \mathbf{U}^i + R\omega^i(n_1^i \mathbf{e}_2 - n_2^i \mathbf{e}_1), \quad \mathbf{g}_j = \mathbf{U}^j + R\omega^j(n_1^j \mathbf{e}_2 - n_2^j \mathbf{e}_1). \quad (2.19)$$

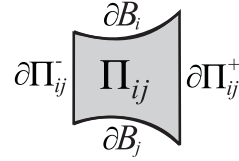


Fig. 2.9. Boundary of the neck Π_{ij}

Due to linearity the minimizer \mathbf{u}_{lub} is decomposed into five vector fields corresponding to the relative motions of inclusions as in Fig. 2.7-2.8 as follows:

$$\begin{aligned} \mathbf{u}_{lub} = & \frac{1}{2}(\mathbf{U}^i + \mathbf{U}^j) + [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij}] \mathbf{u}_1 + [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{q}^{ij}] \mathbf{u}_2 \\ & + R(\omega^i + \omega^j) \mathbf{u}_3 + R(\omega^i - \omega^j) \mathbf{u}_4, \end{aligned} \quad (2.20)$$

where functions \mathbf{u}_k , $k = 1, \dots, 4$ are minimizers of $W_{\Pi_{ij}}$ in \mathcal{V}_{ij} (equation (2.18)) where boundary conditions (2.19) are replaced, respectively, by:

1) the *shear* motion of the fluid between two neighboring inclusions:

$$\mathbf{u}_1|_{\partial B^i} = \mathbf{g}_i = \frac{1}{2} \mathbf{e}_1, \quad \mathbf{u}_1|_{\partial B^j} = \mathbf{g}_j = -\frac{1}{2} \mathbf{e}_1, \quad (2.21)$$

2) the *squeeze* motion of the fluid between neighbors:

$$\mathbf{u}_2|_{\partial B^i} = \mathbf{g}_i = \frac{1}{2} \mathbf{e}_2, \quad \mathbf{u}_2|_{\partial B^j} = \mathbf{g}_j = -\frac{1}{2} \mathbf{e}_2, \quad (2.22)$$

3) the *rotation in the same directions*:

$$\mathbf{u}_3|_{\partial B^i} = \mathbf{g}_i = \frac{1}{2}(n_1^i \mathbf{e}_2 - n_2^i \mathbf{e}_1), \quad \mathbf{u}_3|_{\partial B^j} = \mathbf{g}_j = \frac{1}{2}(n_1^j \mathbf{e}_2 + n_2^j \mathbf{e}_1), \quad (2.23)$$

4) the *rotation in the opposite directions*:

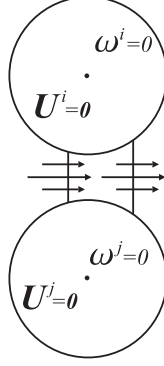


Fig. 2.10. The Poiseuille's microflow between motionless inclusions

$$\mathbf{u}_4|_{\partial B^i} = \mathbf{g}_i = \frac{1}{2}(n_1^i \mathbf{e}_2 - n_2^i \mathbf{e}_1), \quad \mathbf{u}_4|_{\partial B^j} = \mathbf{g}_j = -\frac{1}{2}(n_1^j \mathbf{e}_2 + n_2^j \mathbf{e}_1). \quad (2.24)$$

Let \mathbf{u} be the minimizer of (2.4) then

$$\mathbf{u}_p := \mathbf{u} - \mathbf{u}_{lub} \quad (2.25)$$

minimizes $W_{\Pi_{ij}}$ in the class

$$\begin{aligned} \mathcal{V}_p = \left\{ \mathbf{v} \in H^1(\Pi_{ij}) : \nabla \cdot \mathbf{v} = 0 \text{ in } \Pi_{ij}, \mathbf{v} = \mathbf{0} \text{ on } \partial B^i, \mathbf{v} = \mathbf{0} \text{ on } \partial B^j, \right. \\ \left. \frac{1}{R} \int_{\ell_{ij}} (\mathbf{v} + \mathbf{u}_{lub}) \cdot \mathbf{n} ds = \beta_{ij}^* \right\}. \end{aligned} \quad (2.26)$$

The vector field \mathbf{u}_p describes the flow between two motionless inclusions, hence, it is natural to call it the *Poiseuille microflow*.

Denote by $\boldsymbol{\beta}^* = \{\beta_{ij}^*\}_{i \in \mathbb{I} \cup \mathbb{B}, j \in \mathcal{N}_i}$ and introduce the following set of discrete variables

$$\begin{aligned} \mathcal{R}^* = \{(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}^*) : \mathbf{U}^i, \omega^i \text{ satisfying (2.14) for } i \in \mathbb{B} \\ \text{and } (\mathbb{U}, \boldsymbol{\beta}^*) \text{ satisfying (2.17)}\}. \end{aligned} \quad (2.27)$$

It is straightforward to show that (see Lemma 4.2)

$$\widehat{W}_{\Pi} = \min_{(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}^*) \in \mathcal{R}^*} \sum_{i \in \mathbb{I}, j \in \mathcal{N}_i} \min_{V_{ij}} W_{\Pi_{ij}}(\cdot), \quad (2.28)$$

where V_{ij} is defined by

$$\begin{aligned}
V_{ij} = \{ & \mathbf{v} \in H^1(\Pi_{ij}) : \nabla \cdot \mathbf{v} = 0 \text{ in } \Pi_{ij}, \mathbf{v} = \mathbf{U}^i + R\omega^j(n_1^i \mathbf{e}_2 - n_2^i \mathbf{e}_1) \text{ on } \partial B^i, \\
& \mathbf{v} = \mathbf{U}^j + R\omega^i(n_1^j \mathbf{e}_2 - n_2^j \mathbf{e}_1) \text{ on } \partial B^j \text{ (} j \in \mathbb{I} \text{) or } \mathbf{v} = \mathbf{f} \text{ on } \partial B^j \text{ (} j \in \mathbb{B} \text{)} \\
& \left. \frac{1}{R} \int_{\ell_{ij}} \mathbf{v} \cdot \mathbf{n} ds = \beta_{ij}^* \right\},
\end{aligned} \tag{2.29}$$

with \mathbf{U}^i , ω^i and β_{ij}^* to be components of some no longer arbitrary but fixed $(\mathbb{U}, \omega, \beta^*) \in \mathcal{R}^*$. Functions from V_{ij} are defined in a single neck $\Pi_{ij} \in \Pi$.

By direct computations vector fields

$$\begin{aligned}
\mathbf{u}_t &= \frac{1}{2}(\mathbf{U}^i + \mathbf{U}^j), \\
\mathbf{u}_{sh} &= [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij}] \mathbf{u}_1 + R(\omega^i + \omega^j) \mathbf{u}_3, \\
\mathbf{u}_{sq} &= [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{q}^{ij}] \mathbf{u}_2, \\
\mathbf{u}_{per} &= R(\omega^i - \omega^j) \mathbf{u}_4 + \mathbf{u}_p,
\end{aligned} \tag{2.30}$$

are orthogonal with respect to the scalar product induced by the dissipation functional $W_{\Pi_{ij}}$:

$$\begin{aligned}
W_{\Pi_{ij}}(\mathbf{u}) &= W_{\Pi_{ij}}(\mathbf{u}_{sh}) + W_{\Pi_{ij}}(\mathbf{u}_{sq}) + W_{\Pi_{ij}}(\mathbf{u}_{per}), \quad W_{\Pi_{ij}}(\mathbf{u}_t) = 0, \\
\mathbf{u} &= \mathbf{u}_t + \mathbf{u}_{sh} + \mathbf{u}_{sq} + \mathbf{u}_{per}.
\end{aligned} \tag{2.31}$$

Physically, the decomposition (2.31) corresponds to three well-known types of microflows. Namely, (i) the shear type arises when a pair of inclusions either rotates in the same direction or disks move into opposite directions (Fig. 2.8a, 2.7a), (ii) the squeeze type, when two inclusions in a pair move towards or away each other in thin gaps (Fig. 2.7b), (iii) and permeation of the fluid through the thin gaps between neighbors due to Poiseuille flow between motionless inclusions or rotation of neighbors in the opposite directions (Fig. 2.10, 2.8b).

We are now ready to introduce a quadratic form which determines the discrete dissipation rate. To this end from now on instead of β_{ij}^* as a discrete variable we will use:

$$\beta_{ij} = \beta_{ij}^* - \frac{\delta_{ij}}{2R} [(\mathbf{U}^i + \mathbf{U}^j) \cdot \mathbf{p}^{ij}] - (\omega^i - \omega^j) \frac{\delta_{ij}}{2R} \left[1 + \frac{\delta_{ij}}{4R} \right]. \tag{2.32}$$

The reason for this replacement is that β_{ij} is invariant with respect to Galilean transformation whereas β_{ij}^* is not (see remark in the end of Subsection 1). For example, if a constant vector \mathbf{U}^0 is added to both \mathbf{U}^i and \mathbf{U}^j then the total flux β_{ij}^* changes while β_{ij} stays the same. Also, β_{ij}^* is the total flux through ℓ_{ij} of the entire flow (including the parallel translation, shear, squeeze and permeation) whereas β_{ij} is the flux due to the Poiseuille microflow solely. Finally, the use of β_{ij} simplifies the discrete dissipation form (equation (2.36)).

The use of β_{ij} instead of β_{ij}^* leads to the replacement of the class \mathcal{R}^* (2.27) by \mathcal{R} defined as follows:

$$\mathcal{R} = \{(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}) : \mathbf{U}^i, \omega^i \text{ satisfying boundary condition on } \partial\Omega \text{ (2.14) for } i \in \mathbb{B},$$

$$(\mathbb{U}, \boldsymbol{\beta}) \text{ satisfying weak incompressibility condition (2.17), (2.32)}\}. \quad (2.33)$$

Introduce the *overall discrete dissipation rate*:

$$\mathcal{I} := Q(\widehat{\mathbb{U}}, \widehat{\boldsymbol{\omega}}, \widehat{\boldsymbol{\beta}}) = \min_{(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}) \in \mathcal{R}} Q(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}), \quad (2.34)$$

where

$$Q(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}) = \sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i} Q_{ij}, \quad (2.35)$$

$$\begin{aligned} Q_{ij}(\mathbf{U}^i, \mathbf{U}^j, \omega^i, \omega^j, \beta_{ij}) &= [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij} + R\omega^i + R\omega^j]^2 \mathcal{C}_1^{ij} \delta^{-1/2} \\ &\quad + [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{q}^{ij}]^2 \left(\mathcal{C}_2^{ij} \delta^{-3/2} + \mathcal{C}_3^{ij} \delta^{-1/2} \right) \\ &\quad + \beta_{ij}^2 \left(\mathcal{C}_4^{ij} \delta^{-5/2} + \mathcal{C}_5^{ij} \delta^{-3/2} + \mathcal{C}_6^{ij} \delta^{-1/2} \right) \\ &\quad + R(\omega^i - \omega^j) \beta_{ij} \left(\mathcal{C}_7^{ij} \delta^{-3/2} + \mathcal{C}_8^{ij} \delta^{-1/2} \right) \\ &\quad + R^2(\omega^i - \omega^j)^2 \mathcal{C}_9^{ij} \delta^{-1/2}, \quad \text{for } j \in \mathcal{N}_i \cap \mathbb{I}, \end{aligned} \quad (2.36)$$

$$\begin{aligned} Q_{ij}(\mathbf{U}^i, \omega^i, \beta_{ij}, \mathbf{f}) &= \beta_{ij}^2 \left[\mathcal{B}_1^{ij} \delta^{-5/2} + \mathcal{B}_2^{ij} \delta^{-3/2} + \mathcal{B}_3^{ij} \delta^{-1/2} \right] \\ &\quad + [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij} + R\omega^i]^2 \mathcal{B}_4^{ij} \delta^{-1/2} \\ &\quad + R^2(\omega^i - \omega^j)^2 \mathcal{B}_5^{ij} \delta^{-1/2} \\ &\quad + [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{q}^{ij}]^2 \left(\mathcal{B}_6^{ij} \delta^{-3/2} + \mathcal{B}_7^{ij} \delta^{-1/2} \right) \\ &\quad + \beta_{ij} [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij} + R\omega^i] (\mathcal{B}_8^{ij} \delta^{-3/2} + \mathcal{B}_9^{ij} \delta^{-1/2}) \\ &\quad + \beta_{ij} R(\omega^i - \omega^j) (\mathcal{B}_{10}^{ij} \delta^{-3/2} + \mathcal{B}_{11}^{ij} \delta^{-1/2}) \\ &\quad + \beta_{ij} R\omega^i \mathcal{B}_{12}^{ij} \delta^{-1/2} \\ &\quad + [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij} + R\omega^i] R(\omega^i - \omega^j) \mathcal{B}_{13}^{ij} \delta^{-1/2} \\ &\quad + [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{q}^{ij}] Ra \mathcal{B}_{14}^{ij} \delta^{-1/2}, \quad \text{for } j \in \mathcal{N}_i \cap \mathbb{B}, \end{aligned} \quad (2.37)$$

called the *discrete dissipation rates*, with coefficients \mathcal{C}_k^{ij} , $k = 1, \dots, 9$, \mathcal{B}_m^{ij} , $m = 1, \dots, 14$, which depend on μ , the ratio $\frac{R}{d_{ij}}$ and explicitly given by (A.1) in Appendix 1.

The solution of the discrete problem (2.34) is a set of discrete variables $(\widehat{\mathbb{U}}, \widehat{\boldsymbol{\omega}}, \widehat{\boldsymbol{\beta}}) \in \mathcal{R}$, where $\widehat{\mathbb{U}}$ represents the set of translational velocities of inclusions, $\widehat{\boldsymbol{\omega}}$ the set of angular velocities and $\widehat{\boldsymbol{\beta}}$ characterizes the Poiseuille microflow in necks between neighboring inclusions.

Remark 2.2. $Q(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta})$, defined by (2.35), is a positive definite quadratic form (see Appendix 1).

Remark 2.3. The agreement of the coefficients in (2.36) (explicitly given in (A.1)) with the previous results of [8] is as follows. Only coefficients $\mathcal{C}_1^{ij}, \mathcal{C}_2^{ij}$ coincide with the corresponding coefficients C_{sh}^{ij}, C_{sp}^{ij} in [8]. This is because the coefficients in [8] are obtained by using the approximation of circular surfaces of inclusions by parabolas whereas in this paper we use the true circular surfaces. The main objective of [8] was capturing the strong blow up term of order $\delta^{-3/2}$ only, which requires coefficients C_{sp}^{ij} (\mathcal{C}_2^{ij}). The parabolic approximation does not bring any discrepancy in C_{sp}^{ij} whereas it may bring a discrepancy in some other coefficients. Also since in [8] only the leading term was considered under special boundary conditions there was no need to consider the Poiseuille microflow.

Both Theorems 2.1 and 2.2 are based on the following technical proposition.

Proposition 2.1. *Suppose Ω_F satisfies the close packing condition. Let the triple $(\widehat{\mathbb{U}}, \widehat{\boldsymbol{\omega}}, \widehat{\boldsymbol{\beta}}) \in \mathcal{R}$ solve the discrete problem (2.34)-(2.37). Then the following estimate holds:*

$$|\widehat{W} - \mathcal{I}| \leq \mu \left(\sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i} C_1 \widehat{\beta}_{ij}^2 + C_2 |\widehat{U}^i - \widehat{U}^j|^2 + C_3 \sum_{i \in \mathbb{I} \cup \mathbb{B}} R^2 (\widehat{\omega}^i)^2 \right). \quad (2.38)$$

where $C_k, k = 1, 2, 3$, are dimensionless constants.

The quadratic form Q defined by (2.35)-(2.37) can be written in the following form:

$$Q = Q_{sh}^{in} + Q_{sq}^{in} + Q_{per}^{in} + Q_{per}^b + Q_{sq}^b, \quad (2.39)$$

where

$$\begin{aligned} Q_{sh}^{in} &= \sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i \cap \mathbb{I}} [(U^i - U^j) \cdot p^{ij} + R\omega^i + R\omega^j]^2 \mathcal{C}_1^{ij} \delta^{-1/2}, \\ Q_{sq}^{in} &= \sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i \cap \mathbb{I}} [(U^i - U^j) \cdot q^{ij}]^2 (\mathcal{C}_2^{ij} \delta^{-3/2} + \mathcal{C}_3^{ij} \delta^{-1/2}), \\ Q_{per}^{in} &= \sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i \cap \mathbb{I}} \left\{ \beta_{ij}^2 (\mathcal{C}_4^{ij} \delta^{-5/2} + \mathcal{C}_5^{ij} \delta^{-3/2} + \mathcal{C}_6^{ij} \delta^{-1/2}) \right. \\ &\quad \left. + R(\omega^i - \omega^j) \beta_{ij} (\mathcal{C}_7^{ij} \delta^{-3/2} + \mathcal{C}_8^{ij} \delta^{-1/2}) \right. \\ &\quad \left. + R^2 (\omega^i - \omega^j)^2 \mathcal{C}_9^{ij} \delta^{-1/2} \right\} \end{aligned}$$

$$\begin{aligned}
Q_{per}^b &= \sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i \cap \mathbb{B}} \left\{ \beta_{ij}^2 \left[\mathcal{B}_1^{ij} \delta^{-5/2} + \mathcal{B}_2^{ij} \delta^{-3/2} + \mathcal{B}_3^{ij} \delta^{-1/2} \right] \right. \\
&\quad + \mathcal{B}_8^{ij} \beta_{ij} \left[(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij} + R\omega^i \right] \delta^{-3/2} + \mathcal{B}_{10}^{ij} \beta_{ij} R(\omega^i - \omega^j) \delta^{-3/2} \\
&\quad + \mathcal{B}_9^{ij} \beta_{ij} \left[(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij} + R\omega^i \right] \delta^{-1/2} + \mathcal{B}_{11}^{ij} \beta_{ij} R(\omega^i - \omega^j) \delta^{-1/2} \\
&\quad + \mathcal{B}_{12}^{ij} \beta_{ij} R\omega^i \delta^{-1/2} + \mathcal{B}_4^{ij} \left[(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij} + R\omega^i \right]^2 \delta^{-1/2} \\
&\quad + \mathcal{B}_{13}^{ij} \left[(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij} + R\omega^i \right] R(\omega^i - \omega^j) \delta^{-1/2} \\
&\quad \left. + \mathcal{B}_5^{ij} R^2 (\omega^i - \omega^j)^2 \delta^{-1/2} \right\}, \\
Q_{sq}^b &= \sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i \cap \mathbb{B}} \left\{ \left[(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{q}^{ij} \right]^2 \left(\mathcal{B}_6^{ij} \delta^{-3/2} + \mathcal{B}_7^{ij} \delta^{-1/2} \right) \right. \\
&\quad \left. + \mathcal{B}_{14}^{ij} \left[(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{q}^{ij} \right] R\alpha \delta^{-1/2} \right\}. \tag{2.40}
\end{aligned}$$

Remark 2.4. Both Q_{per}^{in} and Q_{per}^b are quadratic forms (see Appendix I).

Remark 2.5. The discrete dissipation rate \mathcal{I} makes physics transparent. We can see how microflows enter this dissipation rate. Indeed, the *discrete dissipation form* Q is presented as a sum of three motions corresponding to the decomposition (2.31). The decomposition (2.39) reflects the physics of the problem. Namely, for the interior necks the first term Q_{sh}^{in} corresponds to the dissipation rate $W_{\Pi_{ij}}(\mathbf{u}_{sh})$ in (2.31) and describes the shear flow between inclusions. This type of flow is produced by two motions: rotation in the same direction (Fig. 2.8a) and relative shear (Fig. 2.7a). The second term Q_{sq}^{in} corresponds to the dissipation rate $W_{\Pi_{ij}}(\mathbf{u}_{sq})$ in (2.31) and describes the local flow due to the squeeze motion (Fig. 2.7b). The third term Q_{per}^{in} corresponds to the dissipation rate $W_{\Pi_{ij}}(\mathbf{u}_{per})$ in (2.31) and describes the “permeation” type flow due to the Poiseuille microflow between motionless inclusions (Fig. 2.10) and the rotation into the opposite directions (Fig. 2.8b). Finally, the last term Q_b describes the local flow in the thin gap between a disk and the external boundary. Here the decomposition into the parts corresponding to permeation Q_{per}^b and squeeze Q_{sq}^b in boundary necks is similar to one in interior necks.

Remark 2.6. The asymptotics (2.38) is given in terms of the *discrete dissipation rates* Q_{ij} . These dissipation rates are quadratic forms of the key physical parameters $\mathbb{U}, \omega, \beta$. In particular, the overall discrete dissipation rate reveals the functional dependance of the overall properties of suspensions (dissipation rate, effective viscosity, effective permeability) on the microflows (the shear, squeeze and permeation between neighbors).

3 Anomalous Rate of Blow up of the Dissipation Rate, Qualitative Conclusions, Discussions and Open Problems

In this section we show that in 3D the Poiseuille microflow does not occur (the fluid simply flows around the neck). In contrast, in 2D incompressible

fluid may need to permeate through necks. Necks separate fluid domain into *disconnected* regions, triangles, which may have different pressures p_1 , p_2 (Fig. 3.1). In this sense, the 2D problem becomes more complicated than an analogous 3D problem. Moreover, this type of flow in experimental 2D settings may lead to an observable physical effect.

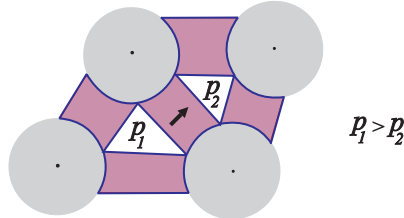


Fig. 3.1. Local flux due to pressure drop

The techniques used in the previous study [8] of 2D flow were capable of capturing only the strong blow up term of order $O(\delta^{-3/2})$ of the asymptotics of the effective viscosity. The subsequent study of [10] reveals the significance of the weak blow up of order $O(\delta^{-1/2})$ in 2D. It was shown that it becomes the leading term in the asymptotics of the shear effective viscosity. The objectives of [10] was to study the degeneracy of the strong blow up term and evaluate the order of the magnitude of the next term which was shown to exhibit the weak blow up. The qualitative conclusion of the analysis of [10] was that the shear viscosity exhibits the weak blow up in both 2D and 3D. While this study highlighted the significance of the weak blow up term, the techniques of [8] did not allow to calculate it. As shown in Theorem 2.1 above the fictitious fluid approach suggested in this paper allows to calculate *any* singular term, and, in particular, the weak blow up. While a generalization of techniques of [8] for 3D suspensions leads to increasing technical difficulties, the fictitious fluid approach provides an appropriate tool to attack this problem. We anticipate that this approach would also be useful in a variety of similar physical problems (e.g. rigid inclusions in an elastic medium in both 2D and 3D). Moreover, the analysis by this approach reveals the significance of the Poiseuille microflows. In this section we present an example which illustrates the following. For the suspensions of free inclusions the Poiseuille microflows contribute to the weak blow up. However, if an external field, which “clamps” inclusions, is imposed on inclusions then the Poiseuille microflow may result in a new type singular behavior of viscous dissipation rate (superstrong blow up). We also present an example of one disk in the fluid that can be clamped by the fluid flow with no external field. However, it is not clear whether this example can be generalized to an ensemble of inclusions. Our example may suggest that on a suspension of free inclusions the superstrong blow up may

occur only due to boundary layer effects and therefore for the large number of inclusions becomes negligible.

1 Suspensions in a Pinning Field

Example 3.1. The techniques described above can be applied to the problems of suspensions of non-neutrally buoyant rigid inclusions defined by

$$\left\{ \begin{array}{ll} (a) \mu \Delta \mathbf{u} = \nabla p, & \mathbf{x} \in \Omega_F \\ (b) \nabla \cdot \mathbf{u} = 0, & \mathbf{x} \in \Omega_F \\ (c) \mathbf{u} = \mathbf{U}^i + R\omega^i(n_1^i \mathbf{e}_2 - n_2^i \mathbf{e}_1), \mathbf{x} \in \partial B^i, & i = 1 \dots N \\ (d) \int_{\partial B^i} \boldsymbol{\sigma}(\mathbf{u}) \mathbf{n}^i ds = m \mathbf{g}, & i = 1 \dots N \\ (e) \int_{\partial B^i} \mathbf{n}^i \times \boldsymbol{\sigma}(\mathbf{u}) \mathbf{n}^i ds = 0, & i = 1 \dots N \\ (f) \mathbf{u} = \mathbf{f}, & \mathbf{x} \in \partial \Omega \end{array} \right. \quad (3.1)$$

where the domain Ω_F is depicted in Fig. 3.2. This problem corresponds to minimization of $W_{\Omega_F}(\mathbf{u}) + \sum_{i=1}^N m \mathbf{g} \cdot \mathbf{U}^i$. The additional term does not change the analysis.

Here we suppose that the density of the solid inclusions is ρ_s and the fluid density is ρ_f . Then the force on the inclusion in the left hand side of (3.1d) counteracts the external gravitational field $-m \mathbf{g}$, where $\mathbf{g} = (0, g)$ is acceleration due to the gravity and $m = \pi R^2(\rho_s - \rho_f)$ is the excess mass. We choose the external boundary condition \mathbf{f} and the force exerted by the heavy disks on the fluid so that the inclusions do not move and fluid is forced to permeate through the thin gaps between motionless inclusions. The force exerted by the disks is equal to their weight $\pi R^2(\rho_s - \rho_f)g$, where we choose inclusions to be superheavy:

$$\rho_s = C\delta^{-5/2}, \text{ for some } C = C(\mu, R) > 0, \quad (3.2)$$

and the applied boundary data \mathbf{f} is chosen so that

$$\mathbf{f} = \begin{pmatrix} 0 \\ f_2(x, y) \end{pmatrix} \in H^{1/2}(\partial \Omega), \quad (3.3)$$

where f_2 is some periodic function of $(x, y) \in \partial \Omega$. Such a boundary data \mathbf{f} and the asymptotics of ρ_s are selected so that inclusions do not move (the gravity balances the viscous force). This balance can be found by solving an auxiliary problem similar to the one considered in [32] (for details see Appendix 4).

Then the following proposition holds.

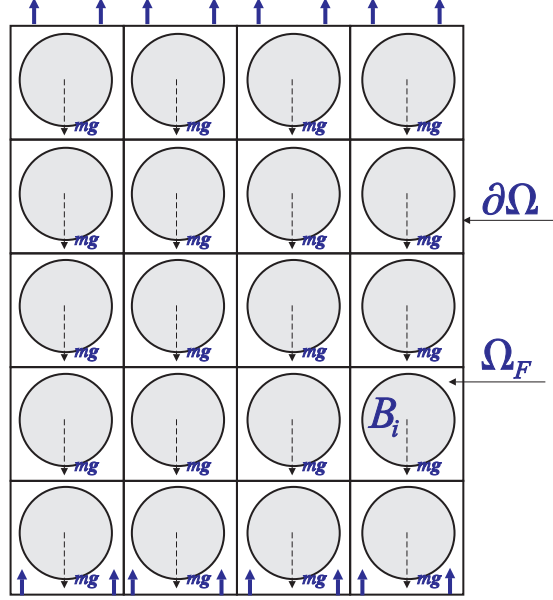


Fig. 3.2. Example of a domain occupied by a suspension in a presence of a pinning field (gravity)

Proposition 3.1 (Superstrong Blow Up due to a Pinning Field). *Let \widehat{W} be the overall viscous dissipation rate of the problem (3.1). There exist ρ_s and \mathbf{f} of the form (3.2) and (3.3), respectively, such that the following asymptotic representation holds:*

$$\widehat{W} = N \left(C_1 \delta^{-5/2} + C_2 \delta^{-3/2} + C_3 \delta^{-1/2} \right) + O(1), \quad \text{as } \delta \rightarrow 0,$$

$$\text{where } C_1 = \frac{9}{4} \pi \mu R^{5/2}, C_2 = \frac{99}{160} \pi \mu R^{3/2}, C_3 = \frac{29241}{17920} \pi \mu R^{1/2},$$

and N is the total number of inclusions.

The proof of this proposition is also given in Appendix 4).

2 No Singularity of the Dissipation Rate due to the Poiseuille Microflow in 3D

Example 3.2. In order to explain why the the Poiseuille microflow does not contribute to the singular behavior of the dissipation rate in 3D let us consider what happens with the fluid between two neighboring inclusions. Let $K = [-L, L]^3$ be a cube. The parts of two neighboring inclusions are modeled by the hemispheres attached to the top and bottom faces of the cube as shown in Fig. 3.3. Consider a 3D analog of the problem (2.1) with boundary condition \mathbf{f}

to be a given constant vector \mathbf{V} on two opposite sides of the lateral boundary and zero vector on the rest of the boundary (Fig. 3.3). Since the effective viscous dissipation rate \widehat{W} is bounded from above by the dissipation rate $W(\mathbf{w})$ for any test function \mathbf{w} it suffices to find \mathbf{w} such that $W(\mathbf{w}) = O(1)$.

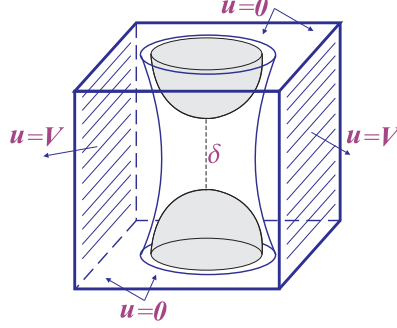


Fig. 3.3. Poiseuille microflow in 3D

Consider a “hourglass domain” Ψ inside of the box containing two hemispheres as in Fig. 3.3. We choose the trial function \mathbf{w} to be:

$$\mathbf{w} = \begin{cases} \mathbf{0}, & \text{when } \mathbf{x} \in \Psi \\ \mathbf{W}, & \text{when } \mathbf{x} \in K \setminus \Psi \end{cases} \quad (3.4)$$

where \mathbf{W} solves the Stokes problem with Dirichlet boundary conditions: $\mathbf{W} = \mathbf{0}$ on the boundary of Ψ and $\mathbf{W} = \mathbf{f}$ on ∂K (such a solution exists and is unique).

Evaluating the dissipation rate on such a trial function we obtain:

$$W(\mathbf{w}) \leq \|\mathbf{w}\|_{H^1(K)}^2 \leq C\|\mathbf{V}\|_{H^{1/2}(\partial K)}^2 = O(1).$$

3 Boundary Layer Effects Leading to Superstrong Blow Up

Example 3.3. Consider the domain $\Omega = (-1, 1)^2$ which contains only one inclusion B . Decompose the domain outside B into necks $\bigcup_{i=1}^4 \Pi_i$ and squares

$\bigcup_{i=1}^4 \square_i$ as in Fig. 3.4(a). Choose

$$\mathbf{f} = \begin{cases} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \zeta_1, & \partial\Box_1 \cap \partial\Omega =: \Gamma_1 \\ \begin{pmatrix} -1 \\ -1 \end{pmatrix} \zeta_2, & \partial\Box_2 \cap \partial\Omega =: \Gamma_2 \\ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \zeta_3, & \partial\Box_3 \cap \partial\Omega =: \Gamma_3 \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \zeta_4, & \partial\Box_4 \cap \partial\Omega =: \Gamma_4 \\ \mathbf{0}, & \text{elsewhere} \end{cases}$$

where ζ_i , $i = 1, \dots, 4$ are smooth function having a compact support outside of Γ_i such that $\zeta_i = 1$ in Γ_i . Moreover, they satisfy the following symmetry condition:

$$\zeta_1(x, y) = \zeta_2(-x, y) = \zeta_3(-x, -y) = \zeta_4(x, -y) \quad \text{and} \quad \zeta_1(x, y) = \zeta_1(y, x).$$

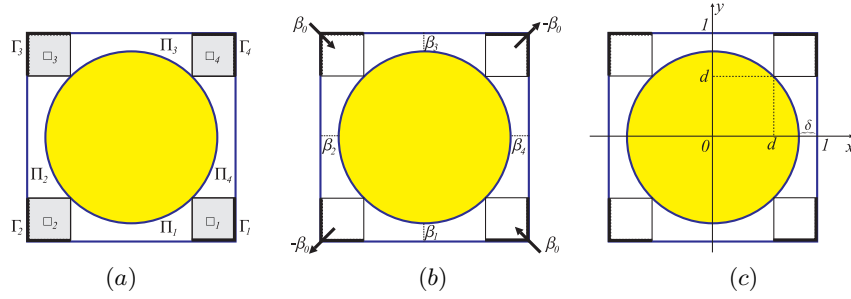


Fig. 3.4. One inclusion example: Boundary layer lead to superstrong blow up of viscous dissipation rate

The fluxes through the parts of the boundary Γ_1 and Γ_3 which are equal and we denote them as

$$\beta_0 = \frac{2}{R}(-1 + d), \quad d = \delta + R(1 - \frac{1}{\sqrt{2}}).$$

Then the fluxes through Γ_2 and Γ_4 are $-\beta_0$.

Due to symmetry of the problem the inclusion does not rotate, that is, $\omega = 0$ and $\beta_1 = \beta_3 = -\beta_2 = -\beta_4$ (see Fig. 3.4(b)). Then

$$\begin{aligned}
\mathcal{I} &= \min_{\mathbf{U}, \beta_i} \sum_{i=1}^4 A\delta^{-3/2}[\mathbf{U} \cdot \mathbf{q}^i]^2 + B\delta^{-1/2}[\mathbf{U} \cdot \mathbf{p}^i]^2 + CR^2\delta^{-5/2}\beta_i^2 \\
&\quad - p_1(\beta_0 + \beta_1 - \beta_4 + U_1 - U_2) - p_2(-\beta_0 + \beta_2 - \beta_1 - U_1 - U_2) \\
&\quad - p_3(\beta_0 + \beta_3 - \beta_2 - U_1 + U_2) - p_4(-\beta_0 + \beta_4 - \beta_3 + U_1 + U_2) \\
&= \min_{\mathbf{U}, \beta_i} \{2A\delta^{-3/2}[U_1^2 + U_2^2] + 2B\delta^{-1/2}[U_1^2 + U_2^2] + 4CR^2\delta^{-5/2}\beta_1^2 \\
&\quad - p_1(2\beta_0 + 4\beta_1) - p_2(-2\beta_0 - 4\beta_1)\},
\end{aligned}$$

where p_i , $i = 1, \dots, 4$ are the Lagrange multipliers corresponding to the weak incompressibility condition (2.17) and $p_1 = p_3$, $p_2 = p_4$.

Solving the Euler-Lagrange equations for this minimization problem we obtain

$$U_1 = U_2 = 0, \quad \beta_1 = -\frac{\beta_0}{2} = \frac{1}{R}(1 - d),$$

which provides the following asymptotics:

$$\mathcal{I} = C(1 - d)^2\delta^{-5/2}.$$

Thus, we see that the superstrong blow up can occur due to the boundary layer where Poiseuille flow is significant.

4 Free Suspensions (no external field)

In the above example 3.1 we demonstrated that the Poiseuille microflow dominates the asymptotics of the overall viscous dissipation rate and may cause the superstrong blow up of order $O(\delta^{-5/2})$. The key ingredient of this example is the presence of a strong external pinning field which “clamps” inclusions or alternatively the presence of the boundary layer as in Example 3.3. In typical suspensions, with no external field, the inclusions are free to move. Then we expect that β is asymptotically small, as $\delta \rightarrow 0$, so that the Poiseuille microflow does not contribute to the superstrong blow up by Theorem 2.1. This observation is also supported by the analysis of the periodicity cell problem with five inclusions in [9]. Below we present an example of a suspension with a hexagonal periodic array of inclusions and prove that for the extensional external boundary conditions the viscous dissipation rate exhibits the strong blow up $O(\delta^{-3/2})$ since all $\beta_{ij} = 0$. Such a hexagonal array in 2D is a representative of a densely packed array of disks.

Example 3.4. Consider a square domain $\Omega = (-M, M)^2$, $M > 0$ with hexagonal array of disks B^i centered at $(x_i, y_i) \in \Omega$. We consider a periodic hexagonal array of such inclusions, as depicted on Figure 3.5).

The boundary value problem (2.1) is supplemented with boundary conditions

$$\mathbf{f} = \begin{pmatrix} x \\ -y \end{pmatrix} \quad \text{on } \partial\Omega. \quad (3.5)$$

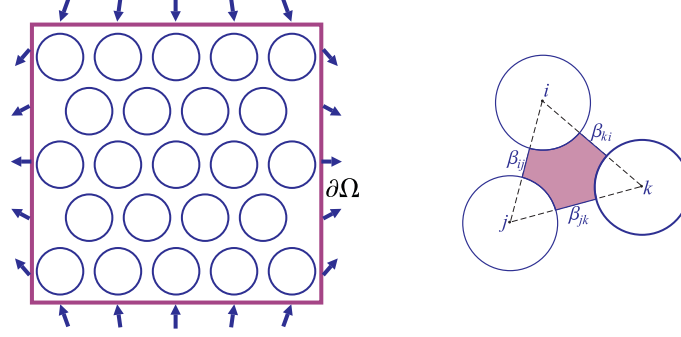


Fig. 3.5. Periodic domain occupied by a suspension under the extensional boundary conditions that exhibits the strong blow up

We first prove that the overall dissipation rate $\widehat{W}_{hex} = O(\delta^{-3/2})$ because we construct a trial vector field \mathbf{v} such that $W_{\Omega_F}(\mathbf{v}) = O(\delta^{-3/2})$. We thus obtain an upper bound for \widehat{W}_{hex} .

The construction is as follows. For each inclusion B^i , centered at $\mathbf{x}_i = (x_i, y_i)$ we prescribe the translational velocity to be exactly

$$\mathbf{U}^i = \begin{pmatrix} x_i \\ -y_i \end{pmatrix}, \quad (3.6)$$

and the rotational velocity $\omega^i = 0$. For such velocities the zero flux constraint (2.17) for every fluid region \mathcal{A}_{ijk} (see the right part of Fig. 3.5) takes the form

$$\beta_{ij} + \beta_{jk} + \beta_{ki} = 0, \quad (3.7)$$

because elementary computations show that for every i, j, k

$$(\mathbf{U}^i + \mathbf{U}^j)\mathbf{p}^{ij} + (\mathbf{U}^j + \mathbf{U}^k)\mathbf{p}^{jk} + (\mathbf{U}^k + \mathbf{U}^i)\mathbf{p}^{ki} = 0.$$

Hence, we can choose all β_{ij} to be identically zero. Then Theorem 2.1 implies that there exists a trial vector field \mathbf{v} such that

$$W_{\Omega_F}(\mathbf{v}) = O(\delta^{-3/2}),$$

and, therefore, $\widehat{W}_{hex} = O(\delta^{-3/2})$.

To show that for this array of inclusions $\widehat{W}_{hex} > C\delta^{-3/2}$ we consider a chain of disks that connects upper and lower boundaries of the domain Ω : $\partial\Omega^+ = \{(x, y) : y = M\}$ and $\partial\Omega^- = \{(x, y) : y = -M\}$, respectively (see Fig. 3.6). We choose this chain so that the y -axis of the coordinate system passes through the centers of disks in this chain. Then

$$\widehat{W} = \min_{\mathcal{R}} Q(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}) \geq (C_2\delta^{-3/2} + C_3\delta^{-1/2}) \min_{\mathbb{U}} \sum_{chain} (U_2^i - U_2^j)^2.$$

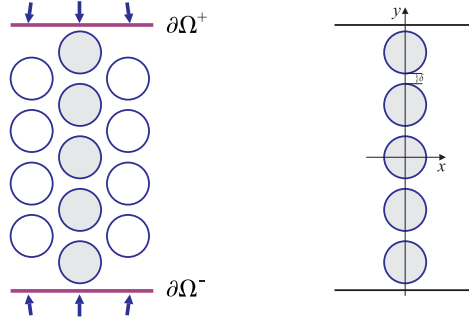


Fig. 3.6. Chain of disks connecting the upper and lower boundaries of Ω

where the last minimum is taken over the disks in the chain (shadowed disks in Fig. 3.6). Then

$$\min_{\mathbb{U}} \sum_{chain} (U_2^i - U_2^j)^2 = A > 0,$$

therefore $\widehat{W}_{hex} > C\delta^{-3/2}$. This leads to a conclusion that the overall viscous dissipation rate \widehat{W}_{per} exhibits the strong blow up of order $\delta^{-3/2}$.

It is known that for local movements of pairs of inclusions the squeeze type motion (Fig. 2.7b) provides the strongest singularity (of order $\delta^{-3/2}$) whereas all other types of motions: rotations and shear provide a weaker singularity (of order $\delta^{-1/2}$). Thus, it is natural to expect that in a suspension where inclusions are free to move with the extensional boundary conditions the viscous dissipation exhibits the singularity of order $\delta^{-3/2}$. The above example shows that this is indeed the case, that is, the superstrong blow up does not occur and the anomalous rate of $O(\delta^{-5/2})$ can be achieved if an external field is applied.

There is also one more case where the superstrong blow up can be obtained. Namely, Example 3.3 shows that it occurs in the boundary layers. Hence the above $O(\delta^{-3/2})$ conclusion applies to bulk effective properties of free suspensions of large number of inclusions, when the boundary effects are negligible.

4 The Fictitious Fluid Problem

The use of the fictitious fluid approach immediately gives a lower estimate on the viscous dissipation rate \widehat{W} as follows.

Lemma 4.1. *For \widehat{W}_{Π} defined by (2.10) and \widehat{W} defined by (2.3) the following inequality holds:*

$$\widehat{W}_{\Pi} \leq \widehat{W}. \quad (4.1)$$

Proof. The idea of the proof is to observe that for any incompressible vector field \mathbf{u} : $W_{\Pi}(\mathbf{u}) \leq W_{\Omega_F}(\mathbf{u})$. Therefore, consider the minimizer \mathbf{u} of (2.4) and

$$\widehat{W} = \mu \int_{\Omega_F} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{u}) \, d\mathbf{x} \geq \mu \int_{\Pi} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{u}) \, d\mathbf{x}. \quad (4.2)$$

since we disregard a part of the domain Ω_F . Hence, $W_{\Pi}(\mathbf{u}) \leq \widehat{W}$.

We only need to show now that the restriction of the minimizer \mathbf{u} on the set Π is an admissible trial field for minimization problem of the fictitious fluid (2.10), that is, $\mathbf{u}|_{\Pi} \in V_{\Pi}$. Indeed, $\mathbf{u}|_{\Pi}$ belongs to $\mathbf{H}^1(\Pi)$, is divergence-free in Π and satisfies

$$0 = \int_{\Delta_{ijk}} \nabla \cdot \mathbf{u} \, d\mathbf{x} = \int_{\partial \Delta_{ijk}} \mathbf{u} \cdot \mathbf{n} \, ds.$$

Thus,

$$\min_{\mathbf{v} \in V_{\Pi}} W_{\Pi}(\mathbf{v}) \leq W_{\Pi}(\mathbf{u}) \leq \widehat{W} = W_{\Omega_F}(\mathbf{u}).$$

□

Another significant advantage of the fictitious fluid problem is that the global minimization problem (2.10) can be split into two consecutive problems: one of them is on a single neck Π_{ij} , and the other one is a minimization problem on discrete variables $(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}^*) \in \mathcal{R}^*$.

Lemma 4.2. (*Iterative minimization lemma*). Suppose \widehat{W}_{Π} is defined by (2.10). Then

$$\widehat{W}_{\Pi} = \min_{(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}^*) \in \mathcal{R}^*} \sum_{i \in \mathbb{I}, j \in \mathcal{N}_i} \min_{V_{ij}} W_{\Pi_{ij}}(\cdot). \quad (4.3)$$

Moreover, the minimizer of $W_{\Pi_{ij}}$ over V_{ij} satisfies the following Euler-Lagrange equations:

$$\left\{ \begin{array}{ll} (a) \, \mu \Delta \mathbf{u} = \nabla p, & \mathbf{x} \in \Pi_{ij}, \\ (b) \, \nabla \cdot \mathbf{u} = 0, & \mathbf{x} \in \Pi_{ij}, \\ (c') \, \mathbf{u} = \mathbf{U}^i + R\omega^i(n_1^i \mathbf{e}_2 - n_2^i \mathbf{e}_1), & \mathbf{x} \in \partial B^i, \\ (c'') \, \mathbf{u} = \mathbf{U}^j + R\omega^j(n_1^j \mathbf{e}_2 - n_2^j \mathbf{e}_1), & \mathbf{x} \in \partial B^j, \\ (d) \, \frac{1}{R} \int_{\ell_{ij}} \mathbf{u} \cdot \mathbf{n} \, ds = \beta_{ij}^*, & \\ (e) \, \boldsymbol{\sigma}(\mathbf{u}) \mathbf{n} = -p_{ij}^{\pm} \mathbf{n}, & \mathbf{x} \in \partial \Pi_{ij}^{\pm}, \\ (f) \, \mathbf{u} = \mathbf{f}, & \mathbf{x} \in \partial \Pi_{ij} \cap \partial \Omega, \end{array} \right. \quad (4.4)$$

where p_{ij}^{\pm} are the Lagrange multipliers for the weak incompressibility condition (2.17).

Proof. Minimizing $W_{\Pi}(\mathbf{u})$ over V_{Π} leads to the Euler-Lagrange equations

$$\left\{ \begin{array}{ll} (a) \mu \Delta \mathbf{u} = \nabla p, & \mathbf{x} \in \Pi \\ (b) \nabla \cdot \mathbf{u} = 0, & \mathbf{x} \in \Pi \\ (c) \mathbf{u} = \mathbf{U}^i + R\omega^i(n_1^i \mathbf{e}_2 - n_2^i \mathbf{e}_1), & \mathbf{x} \in \partial B^i, \quad i = 1 \dots N \\ (d) \int_{\partial B^i} \boldsymbol{\sigma}(\mathbf{u}) \mathbf{n}^i ds = \mathbf{0} & i = 1 \dots N \\ (e) \int_{\partial B^i} \mathbf{n}^i \times \boldsymbol{\sigma}(\mathbf{u}) \mathbf{n}^i ds = \mathbf{0}, & i = 1 \dots N \\ (f) \int_{\partial \Delta_{ijk}} \mathbf{u} \cdot \mathbf{n} ds = 0, & i \in \mathbb{I}, j, k \in \mathcal{N}_i \\ (g) \boldsymbol{\sigma}(\mathbf{u}) \mathbf{n} = p_{ijk} \mathbf{n}, & \mathbf{x} \in \partial \Delta_{ijk} \\ (h) \mathbf{u} = \mathbf{f}, & \mathbf{x} \in \partial \Omega, \end{array} \right. \quad (4.5)$$

where the pressure $p(\mathbf{x})$ arises from the incompressibility condition $\nabla \cdot \mathbf{u} = 0$, $\mathbf{x} \in \Pi$ and “pressure constants” p_{ijk} arise from weak incompressibility condition

$$\int_{\partial \Delta_{ijk}} \mathbf{v} \cdot \mathbf{n} ds = 0.$$

Given the boundary data \mathbf{f} in (4.5) we uniquely determine (see Appendix 3) unknowns

$$\mathbf{u}, p, \mathbf{U}^i, \omega^i, p_{ijk}, \quad i \in \mathbb{I}, j, k \in \mathcal{N}_i. \quad (4.6)$$

Fix a neck Π_{ij} and consider the problem (4.4) on it. For this pair of indices i, j take

$$f, \mathbf{U}^i, \omega^i, \mathbf{U}^j, \omega^j, \beta_{ij}^* = \frac{1}{R} \int_{\ell_{ij}} \mathbf{u} \cdot \mathbf{n} ds \quad (4.7)$$

found from (4.6). Using (4.7) as a given data, boundary value problem (4.4) can be solved uniquely (see Appendix 4). Due to the unique solvability of both (4.5) and (4.4) the pair (\mathbf{u}, p) in (4.6) must solve (4.4) and ⁴

$$p_{ij}^+ = p_{ijk}, p_{ij}^- = p_{ijm},$$

where triangles Δ_{ijk} and Δ_{ijm} are adjacent to the neck Π_{ij} (see Fig.4.1). Hence (4.5) reduces to (4.4).

At this point the completion of the proof would be trivial if we did not have β_{ij}^* . Indeed, in the entire domain Ω_F a result analogous to (4.3) is simply

$$\min_{(\mathbb{U}, \boldsymbol{\omega}, \mathbf{u})} W_{\Omega_F}(\mathbf{u}) = \min_{(\mathbb{U}, \boldsymbol{\omega})} \left(\min_{\mathbf{u}, \text{ when } (\mathbb{U}, \boldsymbol{\omega}) \text{ fixed}} W_{\Omega_F}(\mathbf{u}) \right).$$

Hence it only remains to show that for any given $(\mathbb{U}, \boldsymbol{\omega})$ we can find at least one set $\boldsymbol{\beta}^*$ satisfying the weak incompressibility condition (2.17). To this end we fix

⁴ For notational convenience we identify $p_{ij}^+ = p_{jk}^+ = p_{ki}^+ = p_{ijk}$, and $p_{ij}^- = p_{ji}^-$.

$(\mathbb{U}, \boldsymbol{\omega})$ and let \mathbf{u} be the solution of the Stokes equation $\mu \Delta \mathbf{u} = \nabla p$ and $\nabla \cdot \mathbf{u} = 0$ in the domain Ω_F with the Dirichlet data $\mathbf{u}|_{\partial B^i} = \mathbf{U}^i + R\omega^i(n_1^i \mathbf{e}_2 - n_2^i \mathbf{e}_1)$, $\mathbf{u}|_{\partial\Omega} = \mathbf{f}$. Set $\beta_{ij}^* = \frac{1}{R} \int_{\ell_{ij}} \mathbf{u} \cdot \mathbf{n} ds$. Hence, we obtain $\boldsymbol{\beta}^* = \{\beta_{ij}^*\}$ such that $(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}^*) \in \mathcal{R}^*$. This completes the proof of lemma 4.2.

Remark 4.1. For a given $(\mathbb{U}, \boldsymbol{\omega})$ permeation constants $\boldsymbol{\beta}^*$ may not be unique. In fact, the choice of β_{ij}^* has N degrees of freedom where N is the number of inclusions. Indeed, $\boldsymbol{\beta}^*$ is found from solving a linear system $A\boldsymbol{\beta}^* = \mathbf{b}$ where the number of unknowns equals the number P of interior necks and the number of equations equals the number of triangles, but there are only $P - N$ linearly independent ones. Hence, the number of free parameters is equal to the number of inclusions.

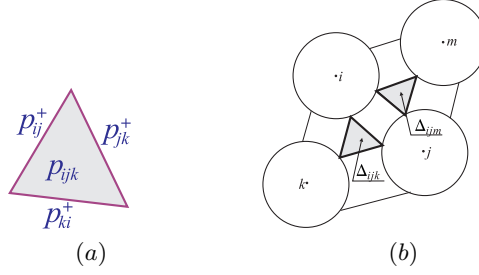


Fig. 4.1. (a) Pressures on the boundary of the triangle Δ_{ijk} ; (b) Two triangles adjacent to the neck Π_{ij}

Remark 4.2. The unknowns of the problem (4.5) are the velocity field $\mathbf{u}(\mathbf{x})$, the pressure $p(\mathbf{x})$, the constant translational \mathbf{U}^i and angular ω^i velocities of the disk B^i , $i = 1, \dots, N$ and constants p_{ijk} . Formally these constants appear as the Lagrange multipliers for the constraints (2.7). Similar to how the pressure $p(\mathbf{x})$ appears as the Lagrange multiplier in the variational formulation corresponding to the Stokes equation, the weak incompressibility condition for the fictitious fluid, inherited from the original fluid, gives rise to the constant Lagrange multipliers, that one can regard as a constant pressure on the boundary of the fictitious fluid domain. This also motivates the notations p_{ijk} . Thus, the fictitious fluid may be also interpreted as follows: an incompressible fluid occupies necks while triangular pockets are filled with “gas” of the constant pressure p_{ijk} . Naturally, the unknowns of the problem (4.4) are the functions $\mathbf{u}(\mathbf{x})$, $p(\mathbf{x})$ and constants p_{ij}^\pm , representing the velocity field, the pressure in the neck Π_{ij} and the constant pressures on the lateral boundary $\partial\Pi_{ij}^\pm$, respectively.

Remark 4.3. A major difficulty in applying previous ‘one-step’ discretization techniques [8, 12, 13] to vectorial problems is the presence of integral constraints in the dual variational formulation. In our ‘two-step’ discretization approach, due to the Iterative Minimization Lemma, the inner minimization problem has Dirichlet boundary conditions on inclusions and therefore neither this problem, nor its dual have any integral conditions. On the other hand, due to (4.3) second minimization implies that these integral conditions are automatically satisfied.

Lemma 4.3. *Suppose Ω_F satisfies the close packing condition. Then for \widehat{W}_Π defined by (2.10) and \widehat{W} defined by (2.3) the following inequality holds:*

$$\widehat{W} \leq \widehat{W}_\Pi + \mu \left(\sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i} C_1 R^2 \widehat{\beta}_{ij}^2 + C_2 |\widehat{\mathbf{U}}^i - \widehat{\mathbf{U}}^j|^2 + C_3 \sum_{i \in \mathbb{I} \cup \mathbb{B}} R^2 (\widehat{\omega}^i)^2 \right), \quad (4.8)$$

where $(\widehat{\mathbf{U}}, \widehat{\omega}, \widehat{\beta}^*)$ minimizes (4.3).

The proof of this lemma relies on the technical construction that appears in Subsection 1 and therefore is put in Subsection 2.

Remark 4.4. Lemma 4.3 is the only place where the close packing condition is necessary to obtain the desired estimate (4.8) because we needed uniform Lipschitz regularity of triangles \triangle_{ijk} .

As a corollary of Lemmas 4.1 and 4.3 we have the main result of this section: the accuracy of approximation of the overall viscous dissipation rate by the dissipation rate of the fictitious fluid given in the following proposition.

Proposition 4.1. *Suppose Ω_F satisfies the close packing condition. Then*

$$|\widehat{W} - \widehat{W}_\Pi| \leq \mu \left(\sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i} C_1 R^2 \widehat{\beta}_{ij}^2 + C_2 |\widehat{\mathbf{U}}^i - \widehat{\mathbf{U}}^j|^2 + C_3 \sum_{i \in \mathbb{I} \cup \mathbb{B}} R^2 (\widehat{\omega}^i)^2 \right), \quad (4.9)$$

where $(\widehat{\mathbf{U}}, \widehat{\omega}, \widehat{\beta}^*)$ minimizes (4.3).

5 Discrete Network

In the previous chapter we described a discrete network that arises from the fictitious fluid approach. Indeed, the equation (4.3) in view of (4.4) is

$$\widehat{W}_\Pi = \min_{(\mathbb{U}, \omega, \beta^*) \in \mathcal{R}^*} \sum_{i \in \mathbb{I}, j \in \mathcal{N}_i} W_{\Pi_{ij}}(\mathbf{u}) =: \min_{(\mathbb{U}, \omega, \beta^*) \in \mathcal{R}^*} \mathcal{W}(\mathbb{U}, \omega, \beta^*), \quad (5.1)$$

where \mathbf{u} is the solution of (4.4) and \mathcal{W} is a positive definite quadratic form of $(\mathbb{U}, \omega, \beta^*)$.

Our next objective is to find coefficients of \mathcal{W} asymptotically as characteristic distance $\delta \rightarrow 0$. We have the following result.

Lemma 5.1. *Suppose $\widehat{W}_{\mathbf{I}}$ is defined by (2.10), \mathcal{I} is defined by (2.34), \mathcal{W} is defined by (5.1), and Q is defined by (2.33)-(2.37). Then*

$$|\mathcal{W}(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}^*) - Q(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta})| \leq \mu \left(\sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i} C_1 R^2 \beta_{ij}^2 + C_2 |\mathbf{U}^i - \mathbf{U}^j|^2 + C_3 \sum_{i \in \mathbb{I} \cup \mathbb{B}} R^2 (\omega^i)^2 \right) \quad (5.2)$$

for any $(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}^*) \in \mathcal{R}^*$ and β_{ij} related to β_{ij}^* through (2.32). In particular,

$$|\widehat{W}_{\mathbf{I}} - \mathcal{I}| \leq \mu \left(\sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i} C_1 R^2 \widehat{\beta}_{ij}^2 + C_2 |\widehat{\mathbf{U}}^i - \widehat{\mathbf{U}}^j|^2 + C_3 \sum_{i \in \mathbb{I} \cup \mathbb{B}} R^2 (\widehat{\omega}^i)^2 \right). \quad (5.3)$$

where $(\widehat{\mathbb{U}}, \widehat{\boldsymbol{\omega}}, \widehat{\boldsymbol{\beta}})$ is the minimizer of Q .

This Lemma shows that coefficients of $\mathcal{W}(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}^*)$ tend to infinity as $\delta \rightarrow 0$ because the corresponding coefficients of $Q(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta})$ are asymptotically large and given in (2.36)-(2.37). The proof of this Lemma is given in Subsection 1.

Combining Proposition 4.1 and Lemma 5.1 we obtain the claim of Proposition 2.1.

In order to prove Theorems 2.1 and 2.2 it remains to show that the error term of the right hand side of (5.3) becomes relatively small compared to the overall discrete dissipation rate \mathcal{I} . In order to show that we prove in the next lemma that $\mathcal{I} \rightarrow \infty$ as $\delta \rightarrow 0$. More specifically we have the following result.

Lemma 5.2. *Suppose Ω_F satisfies the close packing condition. Then there exists a constant $C > 0$ such that for every $\{\mathbf{U}^i\}$:*

$$\begin{aligned} \mathcal{I} \geq \mu \sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i} C_1 \delta^{-3/2} |\mathbf{U}^i - \mathbf{U}^j|^2 + C_2 \delta^{-1/2} R^2 (\omega^i + \omega^j)^2 \\ + C_3 \delta^{-1/2} R^2 (\omega^i - \omega^j)^2 + C_4 \delta^{-5/2} \beta_{ij}^2, \quad \text{as } \delta \rightarrow 0. \end{aligned} \quad (5.4)$$

Proof. We write

$$\begin{aligned} \mathcal{I} = \mu \sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i} C_1 \delta^{-3/2} [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{q}^{ij}]^2 \\ + C_2 \delta^{-1/2} [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij} + R\omega^i + R\omega^j]^2 + C_3 \delta^{-1/2} R^2 (\omega^i - \omega^j)^2 \\ + C_4 \delta^{-5/2} R^2 \beta_{ij}^2. \end{aligned}$$

Using $(a + b)^2 \geq 3/4 a^2 - 3b^2$ we have

$$\begin{aligned} \mathcal{I} \geq \mu \sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i} C_1 \delta^{-3/2} [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{q}^{ij}]^2 - 3C_2 \delta^{-1/2} |\mathbf{U}^i - \mathbf{U}^j|^2 \\ + \frac{3}{4} C_2 \delta^{-1/2} R^2 (\omega^i + \omega^j)^2 + C_3 \delta^{-1/2} R^2 (\omega^i - \omega^j)^2 + C_4 \delta^{-5/2} R^2 \beta_{ij}^2. \end{aligned}$$

By discrete Korn's inequality (Lemma A.1, Appendix 2) when $\delta \rightarrow 0$:

$$C_1 \delta^{-3/2} [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{q}^{ij}]^2 - 3C_2 \delta^{-1/2} |\mathbf{U}^i - \mathbf{U}^j|^2 \geq C \delta^{-3/2} |\mathbf{U}^i - \mathbf{U}^j|^2.$$

Hence, (5.4) holds.

From Lemma 5.2 and Proposition 2.1 we have Theorem 2.2, and Theorem 2.1 follows from Theorem 2.2 and Proposition 4.1.

6 Proofs of Lemmas 5.1 and 4.3

1 Proof of Lemma 5.1

Boundary value problem on a neck. In the local coordinate system the neck Π_{ij} (see Fig.6.1) is given by

$$\Pi_{ij} = \{ \mathbf{x} = (x, y) \in \mathbb{R}^2 : -\gamma_{ij}^- < x < \gamma_{ij}^+, -H_{ij}(x)/2 < y < H_{ij}(x)/2 \}, \quad (6.1)$$

where $|\gamma_{ij}^+ - \gamma_{ij}^-|$ is the neck width, and H_{ij} is the the distance between ∂B_i and ∂B_j defined by

$$H_{ij}(x) = \delta_{ij} + 2R - 2\sqrt{R^2 - x^2}. \quad (6.2)$$

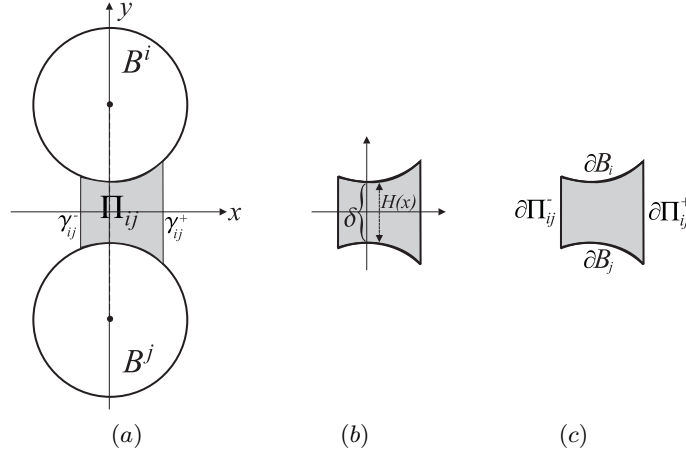


Fig. 6.1. (a) The neck Π_{ij} connecting two neighbors B^i and B^j , (b) the distance $H(x)$ between two neighbors, (c) the boundary $\partial \Pi_{ij}$ of the neck

Fix $(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}^*) \in \mathcal{R}^*$. In the chosen coordinate system the boundary conditions on ∂B^i and ∂B^j for functions defined in (2.30) are written as:

$$\begin{aligned}\mathbf{u}_{sh}|_{\partial B^i} &= \mathbf{G}_1^i = \frac{1}{2}(U_1^i - U_1^j)\mathbf{e}_1 + R(\omega^i + \omega^j) \left(\frac{1}{2}\sqrt{1 - \frac{x^2}{R^2}} \right), \\ \mathbf{u}_{sh}|_{\partial B^j} &= \mathbf{G}_1^j = -\frac{1}{2}(U_1^i - U_1^j)\mathbf{e}_1 + R(\omega^i + \omega^j) \left(-\frac{1}{2}\sqrt{1 - \frac{x^2}{R^2}} \right),\end{aligned}\quad (6.3)$$

$$\begin{aligned}\mathbf{u}_{sq}|_{\partial B^i} &= \mathbf{G}_2^i = \frac{1}{2}(U_1^i - U_1^j)\mathbf{e}_2, \\ \mathbf{u}_{sq}|_{\partial B^j} &= \mathbf{G}_2^j = -\frac{1}{2}(U_1^i - U_1^j)\mathbf{e}_2,\end{aligned}\quad (6.4)$$

$$\begin{aligned}\mathbf{u}_{per}|_{\partial B^j} &= \mathbf{G}_3^i = R(\omega^i - \omega^j) \left(\frac{1}{2}\sqrt{1 - \frac{x^2}{R^2}} \right), \\ \mathbf{u}_{per}|_{\partial B^j} &= \mathbf{G}_3^j = R(\omega^i - \omega^j) \left(-\frac{1}{2}\sqrt{1 - \frac{x^2}{R^2}} \right),\end{aligned}\quad (6.5)$$

and \mathbf{u}_{sh} , \mathbf{u}_{sq} , \mathbf{u}_{per} are minimizers of $W_{\Pi_{ij}}$ over the following sets, respectively:

$$V_{sh} = \left\{ \mathbf{v} \in \mathbf{H}^1(\Pi_{ij}) : \nabla \cdot \mathbf{v} = 0 \text{ in } \Pi_{ij}, \mathbf{v} = \mathbf{G}_1^i \text{ on } \partial B^i, \mathbf{v} = \mathbf{G}_1^j \text{ on } \partial B^j \right\} \quad (6.6)$$

$$V_{sq} = \left\{ \mathbf{v} \in \mathbf{H}^1(\Pi_{ij}) : \nabla \cdot \mathbf{v} = 0 \text{ in } \Pi_{ij}, \mathbf{v} = \mathbf{G}_2^i \text{ on } \partial B^i, \mathbf{v} = \mathbf{G}_2^j \text{ on } \partial B^j \right\} \quad (6.7)$$

$$V_{per} = \left\{ \mathbf{v} \in \mathbf{H}^1(\Pi_{ij}) : \nabla \cdot \mathbf{v} = 0 \text{ in } \Pi_{ij}, \mathbf{v} = \mathbf{G}_3^i \text{ on } \partial B^i, \mathbf{v} = \mathbf{G}_3^j \text{ on } \partial B^j \right. \\ \left. \frac{1}{R} \int_{\ell_{ij}} \mathbf{v} \cdot \mathbf{n} ds = \beta_{ij}^* - \frac{\delta_{ij}}{2R} (U_1^i + U_1^j) \right\}, \quad (6.8)$$

Variational Duality. For any $(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}^*) \in \mathcal{R}^*$ and any $\Pi_{ij} \in \boldsymbol{\Pi}$ we let

$$\mathcal{W}_{ij}(\mathbf{U}^i, \mathbf{U}^j, \omega^i, \omega^j, \beta_{ij}^*) := \min_{V_{ij}} W_{\Pi_{ij}}(\cdot). \quad (6.9)$$

By orthogonality (2.31)

$$\mathcal{W}_{ij}(\mathbf{U}^i, \mathbf{U}^j, \omega^i, \omega^j, \beta_{ij}^*) = W_{\Pi_{ij}}(\mathbf{u}_{sh}) + W_{\Pi_{ij}}(\mathbf{u}_{sq}) + W_{\Pi_{ij}}(\mathbf{u}_{per}).$$

Using variational duality (see e.g. [11, 12])⁵

$$W(\mathbf{u}_{sh}) = \max_{\mathcal{S} \in F} W_1^*(\mathcal{S}), \quad (6.10)$$

$$W(\mathbf{u}_{sq}) = \max_{\mathcal{S} \in F} W_2^*(\mathcal{S}), \quad (6.11)$$

$$W(\mathbf{u}_{per}) = \max_{\mathcal{S} \in F_{per}} W_3^*(\mathcal{S}), \quad (6.12)$$

⁵ Since the construction is done in the neck Π_{ij} hereafter we drop such a subscript in the functional $W_{\Pi_{ij}} = W$. We also drop the indices i, j in $H_{ij}(x) = H(x)$.

where the dual functionals are defined by ($k = 1, 2$)

$$W_k^*(\mathcal{S}) = \int_{\partial B^i} \mathbf{G}_k^i \cdot \mathbf{S} n ds + \int_{\partial B^j} \mathbf{G}_k^j \cdot \mathbf{S} n ds - \frac{1}{4\mu} \int_{\Pi_{ij}} \left[\mathbf{S} : \mathbf{S} - \frac{(\text{tr } \mathbf{S})^2}{2} \right] dx, \quad (6.13)$$

$$\begin{aligned} W_3^*(\mathcal{S}) &= \int_{\partial B^i} \mathbf{G}_3^i \cdot \mathbf{S} n ds + \int_{\partial B^j} \mathbf{G}_3^j \cdot \mathbf{S} n ds \\ &\quad + \frac{R\beta_{ij}^*}{H(\gamma_{ij}^+)} \int_{\partial \Pi_{ij}^+} \mathbf{n} \cdot \mathbf{S} n ds - \frac{R\beta_{ij}^*}{H(\gamma_{ij}^-)} \int_{\partial \Pi_{ij}^-} \mathbf{n} \cdot \mathbf{S} n ds \\ &\quad - \frac{1}{4\mu} \int_{\Pi_{ij}} \left[\mathbf{S} : \mathbf{S} - \frac{(\text{tr } \mathbf{S})^2}{2} \right] dx \end{aligned} \quad (6.14)$$

over the sets

$$F = \left\{ \mathbf{S} \in \mathbb{R}^{2 \times 2}, \mathbf{S} = \mathbf{S}^T, \mathbf{S}_{ij} \in L^2(\Pi_{ij}) : \nabla \cdot \mathbf{S} = \mathbf{0} \text{ in } \Pi_{ij}, \mathbf{S} \mathbf{n} = \mathbf{0} \text{ on } \partial \Pi_{ij}^\pm \right\}. \quad (6.15)$$

$$\begin{aligned} F_{per} &= \left\{ \mathbf{S} \in \mathbb{R}^{2 \times 2}, \mathbf{S} = \mathbf{S}^T, \mathbf{S}_{ij} \in L^2(\Pi_{ij}) : \nabla \cdot \mathbf{S} = \mathbf{0} \text{ in } \Pi_{ij}, \right. \\ &\quad \left. \mathbf{S} \mathbf{n} = \chi_{ij}^\pm \mathbf{n} \text{ on } \partial \Pi_{ij}^\pm \right\}, \end{aligned} \quad (6.16)$$

and χ_{ij}^+ and χ_{ij}^- are arbitrary constants. In (6.14) the numbers $H(\gamma_{ij}^\pm)$ are the lengths of the lateral boundaries, that is, $H(\gamma_{ij}^+) = |\partial \Pi_{ij}^+|$, $H(\gamma_{ij}^-) = |\partial \Pi_{ij}^-|$. It is straightforward to check that the maximizer of problem (6.10), (6.13), (6.15) is the stress tensor $\boldsymbol{\sigma}(\mathbf{u}_{sh})$, the maximizer of (6.11), (6.13), (6.15) is $\boldsymbol{\sigma}(\mathbf{u}_{sq})$ (see e.g. [8]) and the maximizer of (6.12), (6.14), (6.16) is $\boldsymbol{\sigma}(\mathbf{u}_{per})$ (see Appendix 5).

Proof. By direct and dual variational principles for any $\mathbf{S}_1 \in F$, $\mathbf{S}_2 \in F$, $\mathbf{S}_3 \in F_{per}$ and $\mathbf{v}_1 \in V_{sh}$, $\mathbf{v}_2 \in V_{sq}$, $\mathbf{v}_3 \in V_{per}$ we have the following bounds:

$$W_1^*(\mathbf{S}_1) + W_2^*(\mathbf{S}_2) + W_3^*(\mathbf{S}_3) \leq W(\mathbf{u}) \leq W(\mathbf{v}_1) + W(\mathbf{v}_2) + W(\mathbf{v}_3). \quad (6.17)$$

Therefore, if we find trial fields \mathbf{S}_i and \mathbf{v}_i such that the following three inequalities hold (with universal constants \mathcal{C}_i , $i = 1, \dots, 5$)

$$\begin{aligned} |W(\mathbf{v}_1) - W_1^*(\mathbf{S}_1)| &\leq \mu (\mathcal{C}_1[(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij} + R\omega^i + R\omega^j]^2 + \mathcal{C}_2 R^2(\omega^i + \omega^j)^2) \\ &=: E_1(\mathbf{U}^i, \mathbf{U}^j, \omega^i, \omega^j), \end{aligned} \quad (6.18)$$

$$|W(\mathbf{v}_2) - W_2^*(\mathbf{S}_2)| \leq \mu \mathcal{C}_3[(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{q}^{ij}]^2 =: E_2(\mathbf{U}^i, \mathbf{U}^j), \quad (6.19)$$

$$|W(\mathbf{v}_3) - W_3^*(\mathbf{S}_3)| \leq \mu (\mathcal{C}_4 R^2(\omega^i - \omega^j)^2 + \mathcal{C}_5 R^2 \beta_{ij}^2) =: E_3(\omega^i, \omega^j, \beta_{ij}), \quad (6.20)$$

then for $\mathbf{v} = \mathbf{u}_t + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ we have

$$\begin{aligned} |W(\mathbf{u}) - W(\mathbf{v})| &\leq |W(\mathbf{v}_1) - W_1^*(\mathbf{S}_1)| + |W(\mathbf{v}_2) - W_2^*(\mathbf{S}_2)| + |W(\mathbf{v}_3) - W_3^*(\mathbf{S}_3)| \\ &\leq \mu \mathcal{C}_1 \beta_{ij}^2 + \mathcal{C}_2 |\mathbf{U}^i - \mathbf{U}^j|^2 + \mathcal{C}_3 R^2(\omega^i)^2 + \mathcal{C}_4 R^2(\omega^j)^2, \end{aligned}$$

and Lemma 5.1 will follow from summation of the above inequality over all necks and explicit evaluation of $W(\mathbf{v}_1)$, $W(\mathbf{v}_2)$ and $W(\mathbf{v}_3)$.

The construction of \mathbf{S}_i and \mathbf{v}_i for the interior necks is done, for simplicity, for symmetric necks ($\gamma_{ij}^+ = \gamma_{ij}^- =: \gamma_{ij}$) and it is as follows.

$$\begin{aligned} \mathbf{v}_1 = & (U_1^i - U_1^j + R\omega^i + R\omega^j) \begin{pmatrix} yG(x) \\ F(x) - \frac{y^2}{2}G'(x) \end{pmatrix} \\ & + R(\omega^i + R\omega^j) \begin{pmatrix} yK(x) \\ M(x) - \frac{y^2}{2}K'(x) \end{pmatrix} \quad \text{in } \Pi_{ij}, \end{aligned} \quad (6.21)$$

where functions F , G , K and M are found so that \mathbf{v}_1 satisfies (6.3) on $\partial B^i, \partial B^j$, that is,

$$\begin{aligned} G(x) &= \frac{1}{H(x)}, & F(x) &= -\frac{H'(x)}{8}, \\ K(x) &= \frac{\sqrt{1 - \frac{x^2}{R^2}} - 1}{H(x)}, & M(x) &= -\frac{x\delta}{8R\sqrt{R^2 - x^2}}. \end{aligned} \quad (6.22)$$

$$\mathbf{S}_1 = \mu \begin{pmatrix} 0 & G(x) - C \\ G(x) - C & -yG'(x) \end{pmatrix} \quad \text{in } \Pi_{ij}, \quad (6.23)$$

where the constant C is chosen so that $\mathbf{S}_1 \in F$, namely $C = \frac{1}{H(\gamma_{ij})}$.

$$\mathbf{v}_2(x, y) = (U_2^i - U_2^j) \begin{pmatrix} G(x) + 3y^2F(x) \\ -yG'(x) - y^3F'(x) \end{pmatrix} \quad \text{in } \Pi_{ij}, \quad (6.24)$$

where $G(x)$ and $F(x)$ are chosen so that \mathbf{v}_2 to satisfy (6.4) on $\partial B^i, \partial B^j$, that is,

$$G(x) = -\frac{3x}{2H(x)}, \quad F(x) = \frac{2x}{H^3(x)}. \quad (6.25)$$

$$\begin{aligned} (\mathbf{S}_2)_{11} &= \mu(U_2^i - U_2^j) \left(G' + 9y^2F' - 6 \int_{\gamma_0}^x F - C_1 \frac{x^2}{2} - \frac{3}{2}C_2x^2y^2 + C_3 + C_4y^2 \right) \\ (\mathbf{S}_2)_{12} &= (\mathbf{S}_2)_{21} = \mu(U_2^i - U_2^j) (6yF - yG'' - 3y^3F'' + C_1xy + C_2xy^3) \\ (\mathbf{S}_2)_{22} &= \mu(U_2^i - U_2^j) \left(-3G' - 3y^2F' - 6 \int_{\gamma_0}^x F + \frac{y^2}{2}G''' + \frac{3}{4}y^4F''' - C_1\frac{y^2}{2} - C_2\frac{y^4}{4} \right) \end{aligned} \quad (6.26)$$

where constants C_i ($i = 1..4$) are chosen so that $\mathbf{S}_2\mathbf{n} = \mathbf{0}$ on $\partial\Pi_{ij}^\pm$.

$$\mathbf{v}_3 = R(\omega^i - \omega^j) \begin{pmatrix} P(x) + 3y^2Q(x) \\ -yP'(x) - y^3Q'(x) \end{pmatrix} + \beta_{ij}R \begin{pmatrix} K(x) + 3y^2M(x) \\ -yK(x) - y^3M'(x) \end{pmatrix}, \quad (6.27)$$

where functions P , Q , K and M are found so that \mathbf{v}_3 satisfies (6.5) on $\partial B^i, \partial B^j$, that is,

$$\begin{aligned} P(x) &= -\frac{3x^2}{4RH(x)} + \frac{3H(x)}{16R} + \frac{1}{2}\sqrt{1 - \frac{x^2}{R^2}}, \quad Q(x) = \frac{x^2}{RH^3(x)} - \frac{1}{4RH(x)}, \\ K(x) &= \frac{3}{2H(x)}, \quad M(x) = -\frac{2}{H^3(x)}. \end{aligned} \quad (6.28)$$

$$\begin{aligned} (\mathcal{S}_3)_{11} &= \mu \left(P' + 9y^2 Q' - 6 \int Q - C_1 \frac{x^2}{2} - \frac{3}{2} C_2 x^2 y^2 + C_3 + C_4 y^2 \right), \\ (\mathcal{S}_3)_{12} &= (\mathcal{S}_3)_{21} = \mu (6yQ - yP'' - 3y^3 Q'' + C_1 xy + C_2 xy^3), \\ (\mathcal{S}_3)_{22} &= \mu \left(-3P' - 3y^2 Q' - 6 \int Q + \frac{y^2}{2} P''' + \frac{3}{4} y^4 Q''' - C_1 \frac{y^2}{2} - C_2 \frac{y^4}{4} \right), \end{aligned} \quad (6.29)$$

where

$$G(x) = R(\omega^i - \omega^j)P(x) + \beta_{ij}RK(x), \quad F(x) = R(\omega^i - \omega^j)Q(x) + \beta_{ij}RM(x), \quad (6.30)$$

and constants C_i ($i = 1..4$) are chosen so that $\mathcal{S}_3 \mathbf{n} = \chi^\pm \mathbf{n}$ on $\partial \Pi_{ij}^\pm$ for some constants χ^\pm .

In Appendix 1 we verify that the chosen trial fields indeed satisfy inequalities (6.18), (6.19), (6.20), and also show that

$$\begin{aligned} W(\mathbf{v}_1) &= \frac{1}{2} \pi \mu R^{1/2} (U_1^i - U_1^j + R\omega^i + R\omega^j)^2 \delta_{ij}^{-1/2} \\ &\quad + \mathbf{C}_1 \mu (U_1^i - U_1^j + R\omega^i + R\omega^j)^2 + \mathbf{C}_2 \mu R^2 (\omega^i + \omega^j)^2 \\ &\quad + \mathbf{C}_3 \mu (U_1^i - U_1^j + R\omega^i + R\omega^j) R^2 (\omega^i + \omega^j), \end{aligned} \quad (6.31)$$

$$W(\mathbf{v}_2) = (U_2^i - U_2^j)^2 \left[\frac{3}{8} \pi \mu R^{3/2} \delta_{ij}^{-3/2} + \frac{207}{320} \pi \mu R^{-1/2} \delta_{ij}^{-1/2} \right] + \mathbf{C}_4 \mu (U_2^i - U_2^j)^2, \quad (6.32)$$

and

$$\begin{aligned} W(\mathbf{v}_3) &= \beta_{ij}^2 \left(\frac{9}{4} \pi \mu R^{5/2} \delta_{ij}^{-5/2} + \frac{99}{160} \pi \mu R^{3/2} \delta_{ij}^{-3/2} + \frac{29241}{17920} \pi \mu R^{1/2} \delta_{ij}^{-1/2} \right) \\ &\quad + R^2 (\omega^i - \omega^j) \beta_{ij} \left(-3 \pi \mu R^{3/2} \delta_{ij}^{-3/2} + \frac{9}{40} \pi \mu R^{1/2} \delta_{ij}^{-1/2} \right) \\ &\quad + R^2 (\omega^i - \omega^j)^2 \frac{3}{2} \pi \mu R^{1/2} \delta_{ij}^{-1/2} \\ &\quad + \mathbf{C}_5 R^2 (\omega^i - \omega^j)^2 + \mathbf{C}_6 R^2 (\omega^i - \omega^j) \beta_{ij} + \mathbf{C}_7 R^2 \beta_{ij}^2. \end{aligned} \quad (6.33)$$

Coefficients of the quadratic forms Q_{ij} (2.36) can be read off from the formulas (6.31)- (6.33). Construction for the boundary necks and computation of the coefficients of Q_{ij} (2.37) is identical and it is given in Appendix 2.

Remark 6.1. Note that the first term of the decomposition (6.27) indicates the contribution from the fluid motion due to the rotation of neighboring disks in opposite directions while the second term describes the velocity of the fluid between motionless inclusions that is the Poiseuille microflow in a neck. Constants β_{ij} , called *permeation constants*, characterizing this microflow, are the local fluxes through the line segment ℓ_{ij} connecting two motionless inclusions and given by (2.32).

2 Proof of Lemma 4.3

In the previous subsection we showed that for any $(\mathbb{U}, \omega, \beta^*) \in \mathcal{R}^*$ there exists a trial vector field \mathbf{v}_0 on the fictitious fluid domain such that

$$\mathbf{v}_0|_{\partial B^i} = \mathbf{U}^i + R\omega^i(n_1^i \mathbf{e}_2 - n_2^i \mathbf{e}_1), \quad \mathbf{v}_0|_{\partial\Omega} = \mathbf{f},$$

and for any neck Π_{ij}

$$0 \leq W_{\Pi_{ij}}(\mathbf{v}_0) - W_{\Pi_{ij}}(\mathbf{u}) \leq C_1|\mathbf{U}^i - \mathbf{U}^j|^2 + C_2R^2(\omega^i)^2 + C_3R^2(\omega^j)^2 + C_4R^2\beta_{ij}^2, \quad (6.34)$$

where \mathbf{u} is the minimizer of (2.10).

Hence, in order to prove Lemma 5.1 it suffices to show that we can extend \mathbf{v}_0 in every triangle Δ_{ijk} continuously so that this extension, called \mathbf{w} , satisfies

$$\begin{aligned} W_{\Delta_{ijk}}(\mathbf{w}) &\leq C_1|\mathbf{U}^i - \mathbf{U}^j|^2 + C_2|\mathbf{U}^j - \mathbf{U}^k|^2 + C_3|\mathbf{U}^k - \mathbf{U}^i|^2 \\ &\quad + C_4R^2(\omega^i)^2 + C_5R^2(\omega^j)^2 + C_6R^2(\omega^k)^2 \\ &\quad + C_7R^2\beta_{ij}^2 + C_8R^2\beta_{jk}^2 + C_9R^2\beta_{ki}^2, \end{aligned}$$

in every triangle Δ_{ijk} . We construct this extension by assuming that \mathbf{w} solves the Stokes equation on Δ_{ijk} ⁶:

$$\begin{cases} (a) \mu\Delta\mathbf{w} = \nabla p, & \text{in } \Delta_{ijk} \\ (b) \nabla \cdot \mathbf{w} = 0, & \text{in } \Delta_{ijk} \\ (c) \mathbf{w} = \mathbf{v}_0, & \text{on } \partial\Delta_{ijk} \end{cases} \quad (6.35)$$

For (6.35) by [26] we have

$$W_{\Delta_{ijk}} \leq C\|\mathbf{v}_0 - \mathbf{U}\|_{H^{1/2}(\partial\Delta_{ijk})}^2, \quad (6.36)$$

where \mathbf{U} is any constant vector. The constant C may in principle depend on the Lipschitz constant of Δ_{ijk} , but, since the close packing condition is assumed, angles in all Δ_{ijk} are bounded uniformly from 0 and π , and therefore, C in (6.36) is universal.

Let $\mathbf{U} = 1/3(\mathbf{U}^i + \mathbf{U}^j + \mathbf{U}^k)$. Then $(\mathbf{v}_0 - \mathbf{U})$ on $\partial\Delta_{ijk}$ is explicitly given. Namely,

⁶ Here we discuss interior triangles, estimates for the triangles at the boundary are identical.

$$\begin{aligned}
(\mathbf{v}_0 - \mathbf{U})|_{\partial\Pi_{ij}^+} &= \frac{1}{6}(\mathbf{U}^i - \mathbf{U}^k) + \frac{1}{6}(\mathbf{U}^j - \mathbf{U}^k) + [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij} + R\omega^i + R\omega^j] \mathbf{v}_1^1 \\
&\quad + R(\omega^i + \omega^j) \mathbf{v}_1^2 + [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{q}^{ij}] \mathbf{v}_2 + R(\omega^i - \omega^j) \mathbf{v}_3^1 + \beta_{ij} \mathbf{v}_3^2,
\end{aligned}$$

where $\partial\Delta_{ijk} = \partial\Pi_{ij}^+ \cup \partial\Pi_{jk}^+ \cup \partial\Pi_{ki}^+$ (see Fig. 6.2) and $\mathbf{v}_1^1, \mathbf{v}_1^2, \mathbf{v}_2, \mathbf{v}_3^1, \mathbf{v}_3^2$ are polynomial vector fields independent of $(\mathbb{U}, \boldsymbol{\omega}, \boldsymbol{\beta}^*)$ and δ , and given in (6.21), (6.24), (6.27), and β_{ij} is defined by (B.16). Therefore,

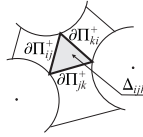


Fig. 6.2. Boundary of the triangle Δ_{ijk}

$$\begin{aligned}
\|\mathbf{v}_0 - \mathbf{U}\|_{H^{1/2}(\partial\Delta_{ijk})}^2 &\leq C_1|\mathbf{U}^i - \mathbf{U}^j|^2 + C_2|\mathbf{U}^j - \mathbf{U}^k|^2 + C_3|\mathbf{U}^k - \mathbf{U}^i|^2 \\
&\quad + C_4R^2(\omega^i)^2 + C_5R^2(\omega^j)^2 + C_6R^2(\omega^k)^2 \\
&\quad + C_7R^2\beta_{ij}^2 + C_8R^2\beta_{jk}^2 + C_9R^2\beta_{ki}^2.
\end{aligned} \tag{6.37}$$

Combining (6.37) with (6.36) we obtain

$$\begin{aligned}
\sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i} W_{\Delta_{ijk}}(\mathbf{w}) &\leq \sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i} C_1|\mathbf{U}^i - \mathbf{U}^j|^2 + C_2|\mathbf{U}^j - \mathbf{U}^k|^2 + C_3|\mathbf{U}^k - \mathbf{U}^i|^2 \\
&\quad + C_4R^2(\omega^i)^2 + C_5R^2(\omega^j)^2 + C_6R^2(\omega^k)^2 \\
&\quad + C_7R^2\beta_{ij}^2 + C_8R^2\beta_{jk}^2 + C_9R^2\beta_{ki}^2.
\end{aligned}$$

Finally, the above inequality and (6.34) give the desired result (4.8).

7 Conclusions

In this paper the asymptotic formula for the overall viscous dissipation rate is obtained, where *all* singular terms are derived and justified. This is done by developing a new technical tool - the two-step fictitious fluid approach. Such approach is expected to be helpful in evaluation of effective properties of various highly packed particulate composites. The obtained asymptotics provides for a complete picture of microflows, while previous studies, mentioned in Introduction, gave only partial analysis of microflows and singular terms. In particular, a new term due to the Poiseuille microflow, which was not taken into account previously, is obtained. It is shown that this Poiseuille microflow does not contribute to singular behavior of viscous dissipation rate in 3D. In contrast, in 2D it may result in an anomalous rate of blow-up (of

order $\delta^{-5/2}$). Indeed, such a rate of blow-up is obtained in the presence of external field (e.g. gravity) or due to the boundary layer effects. Our analysis suggests that in absence of an external field the anomalous blow-up does not occur.

The obtained asymptotics expresses the *continuum* dissipation rate in terms of a *discrete* dissipation rate, and the latter reveals dependence on the key physical parameters.

We remark that this work leads to a somewhat surprising observation that suspensions are actually harder to analyze in 2D than in 3D. As we mentioned above, the Poiseuille type microflow is significant in 2D and it is negligible in 3D. The key reason here is topological: in 2D thin gaps between closely spaced inclusions partition the fluid domain into disconnected regions, which is not the case in 3D. Hence, in 2D permeation of fluid between two inclusions contributes into the singular behavior of the overall viscous dissipation rate.

Finally, we note that 2D mathematical models were often used to analyze qualitative behavior of 3D problems in order to reduce the analytical and computational complexity of the problem. Our work clearly shows limits of validity of such modeling.

A Appendix

1 Coefficients of the quadratic form Q

The coefficients \mathcal{C}_k^{ij} , $k = 1, \dots, 9$, and \mathcal{B}_m^{ij} , $m = 1, \dots, 14$ appearing in (2.36) and (2.37), respectively are given by

$$\begin{aligned}
 \mathcal{C}_1^{ij} &= \frac{1}{2}\pi\mu \left(\frac{R}{d_{ij}}\right)^{1/2}, & \mathcal{C}_2^{ij} &= \frac{3}{4}\pi\mu \left(\frac{R}{d_{ij}}\right)^{3/2}, & \mathcal{C}_3^{ij} &= \frac{207}{320}\pi\mu \left(\frac{R}{d_{ij}}\right)^{1/2}, \\
 \mathcal{C}_4^{ij} &= \frac{9}{4}\pi\mu \left(\frac{R}{d_{ij}}\right)^{5/2}, & \mathcal{C}_5^{ij} &= \frac{99}{160}\pi\mu \left(\frac{R}{d_{ij}}\right)^{3/2}, & \mathcal{C}_6^{ij} &= \frac{29241}{17920}\pi\mu \left(\frac{R}{d_{ij}}\right)^{1/2}, \\
 \mathcal{C}_7^{ij} &= -3\pi\mu \left(\frac{R}{d_{ij}}\right)^{3/2}, & \mathcal{C}_8^{ij} &= \frac{9}{40}\pi\mu \left(\frac{R}{d_{ij}}\right)^{1/2}, & \mathcal{C}_9^{ij} &= \frac{3}{2}\pi\mu \left(\frac{R}{d_{ij}}\right)^{1/2}, \\
 \mathcal{B}_1^{ij} &= 18\pi\mu \left(\frac{R}{d_{ij}}\right)^{5/2}, & \mathcal{B}_2^{ij} &= \frac{51}{20}\pi\mu \left(\frac{R}{d_{ij}}\right)^{3/2}, & \mathcal{B}_3^{ij} &= \frac{20889}{2240}\pi\mu \left(\frac{R}{d_{ij}}\right)^{1/2}, \\
 \mathcal{B}_4^{ij} &= 4\pi\mu \left(\frac{R}{d_{ij}}\right)^{1/2}, & \mathcal{B}_5^{ij} &= \frac{9}{2}\pi\mu \left(\frac{R}{d_{ij}}\right)^{1/2}, & \mathcal{B}_6^{ij} &= 6\pi\mu \left(\frac{R}{d_{ij}}\right)^{3/2}, \\
 \mathcal{B}_7^{ij} &= \frac{63}{20}\pi\mu \left(\frac{R}{d_{ij}}\right)^{1/2}, & \mathcal{B}_8^{ij} &= 6\pi\mu \left(\frac{R}{d_{ij}}\right)^{3/2}, & \mathcal{B}_9^{ij} &= \frac{19}{20}\pi\mu \left(\frac{R}{d_{ij}}\right)^{1/2}, \\
 \mathcal{B}_{10}^{ij} &= -3\pi\mu \left(\frac{R}{d_{ij}}\right)^{3/2}, & \mathcal{B}_{11}^{ij} &= -\frac{3}{8}\pi\mu \left(\frac{R}{d_{ij}}\right)^{1/2}, & \mathcal{B}_{12}^{ij} &= -3\pi\mu \left(\frac{R}{d_{ij}}\right)^{1/2}, \\
 \mathcal{B}_{13}^{ij} &= -3\pi\mu \left(\frac{R}{d_{ij}}\right)^{1/2}, & \mathcal{B}_{14}^{ij} &= 6\pi\mu \left(\frac{R}{d_{ij}}\right)^{1/2}.
 \end{aligned} \tag{A.1}$$

2 Discrete Korn Inequality

Lemma A.1 (Discrete Korn's Inequality). *Suppose Ω_F satisfies the close packing condition. Given $\{\mathbf{U}^i, i \in \mathbb{B}\}$ there exists a constant $C > 0$ such that for every $\{\mathbf{U}^i, i \in \mathbb{I}\}$ the following inequality holds:*

$$\sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i} [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{q}^{ij}]^2 \geq C \sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i} (\mathbf{U}^i - \mathbf{U}^j)^2. \quad (\text{A.2})$$

Proof. By contradiction we assume that there exists $\{\mathbf{U}_k^i\}$ such that

$$\sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i} [(\mathbf{U}_k^i - \mathbf{U}_k^j) \cdot \mathbf{q}^{ij}]^2 < \frac{1}{k} \sum_{i \in \mathbb{I}} \sum_{j \in \mathcal{N}_i} (\mathbf{U}_k^i - \mathbf{U}_k^j)^2.$$

Then we have two options here.

Case 1. There exists a constant $M > 0$ such that for all $i \in \mathbb{I}$ and $k \in \mathbb{N}$: $|\mathbf{U}_k^i| < M$.

Case 2. There exists $i_0 \in \mathbb{I}$ and a subsequence $\{k_n\}$ such that $\mathbf{U}_{k_n}^{i_0} \rightarrow \infty$ as $k_n \rightarrow \infty$. Without loss of generality we set: $|\mathbf{U}_{k_n}^{i_0}| = \max_{i \in \mathbb{I}} |\mathbf{U}_{k_n}^i|$.

First we prove the Case 1. By Bolzano-Weierstrass lemma there exists a subsequence $\{k_n\}$ and a set of vectors $\{\mathbf{V}^i, i \in \mathbb{I}\}$ such that $\mathbf{U}_{k_n}^i \rightarrow \mathbf{V}^i$ as $k_n \rightarrow \infty$ for $i \in \mathbb{I}$ and for \mathbf{V}^i we have:

$$(a_1) (\mathbf{V}^i - \mathbf{V}^j) \cdot \mathbf{q}^{ij} = 0, i, j \in \mathbb{I} \cup \mathbb{B};$$

$$(b_1) \mathbf{V}^i = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, i \in \mathbb{B}.$$

Recall that in this paper we focus on the quasi-hexagonal array of inclusions which triangulates the domain Ω and in which every disk typically has six neighbors. For such an array the condition (a_1) implies that all disks, including quasi-disks, are moving as a rigid body, that is, translate and rotate as a whole. Hence, there exist numbers α, β, φ such that

$$\mathbf{V}^i = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \quad (\text{A.3})$$

However, (b_1) implies that $\alpha = \beta = 0, \cos \varphi = a = 0, b = -s = \sin \varphi$ which are excluded by our boundary conditions (2.2).

For the Case 2 denote by $a_{k_n} := |\mathbf{U}_{k_n}^{i_0}| = \max_{i \in \mathbb{I}} |\mathbf{U}_{k_n}^i| \rightarrow \infty$ as $k_n \rightarrow \infty$.

Consider $\mathbf{V}_{k_n}^i = \frac{\mathbf{U}_{k_n}^i}{a_{k_n}}$. We know that $|\mathbf{V}_{k_n}^i| \leq 1, |\mathbf{V}_{k_n}^{i_0}| = 1$. Hence, there exists a subsequence $\{k_n\}$ and a set of vectors $\{\mathbf{V}^i, i \in \mathbb{I}\}$ such that

$$(a_2) \mathbf{V}_{k_n}^i \rightarrow \mathbf{V}^i \text{ as } k_n \rightarrow \infty;$$

$$(b_2) |\mathbf{V}^{i_0}| = 1;$$

$$(c_2) \mathbf{V}^i = \mathbf{0}, \text{ for } i \in \mathbb{B};$$

$$(d_2) (\mathbf{V}^i - \mathbf{V}^j) \cdot \mathbf{q}^{ij} = 0, i, j \in \mathbb{I} \cup \mathbb{B}.$$

As before, the condition (d_2) implies that all disks move as a rigid body, that is, (A.3) holds. Using (c_2) we have that $\alpha = \beta = \varphi = 0$, hence $\mathbf{V}^i = \mathbf{0}$ for $i \in \mathbb{I} \cup \mathbb{B}$ which contradicts (b_2) .

3 Construction of the network

Here we describe how the network \mathcal{G} is constructed.

We follow [12] where the notion of neighboring inclusions were defined via the *Voronoi tessellation (diagram)* [5] of the domain Ω_F (Fig.A.1). For an arbitrary distribution of the centers \mathbf{x}_i of inclusions B^i , called *sites* or *vertices*:

$$X = \{\mathbf{x}_i \in \Omega, i = 1, \dots, N\},$$

the Voronoi diagram is the partition of the domain Ω into non-overlapping

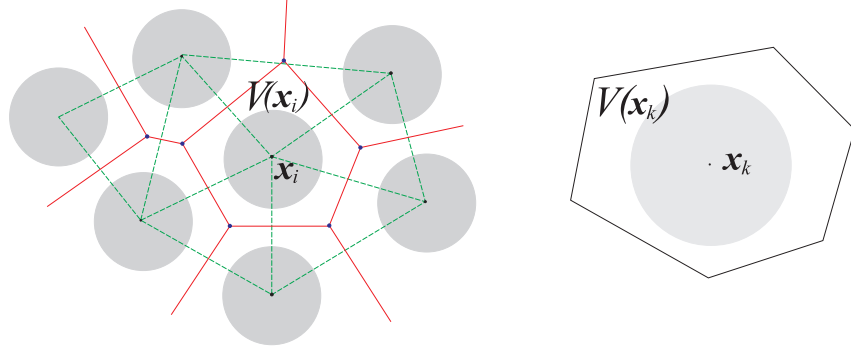


Fig. A.1. Voronoi diagram and Voronoi cell of the site \mathbf{x}_k

Voronoi cells $V(\mathbf{x}_i)$. Each $V(\mathbf{x}_i)$ is the set of all points in Ω that are closer to \mathbf{x}_i than to any other site from X . The situation when site \mathbf{x}_k is located near one of boundaries of Ω is treated similarly to the one in [12, 13]. Namely, one can consider the reflection $\mathbf{x}_{k'}$ of \mathbf{x}_k along this boundary and construct the corresponding Voronoi cell using the obtained auxiliary site $\mathbf{x}_{k'}$ outside of Ω (see Fig.A.2, where the hatched region is a neck), introducing a new site $\mathbf{x}_{k'}$ on the boundary, which is the center of a line segment $B^{k'}$ in $\partial\Omega$, called a *quasidisk* (see also [12, 13]). Two disks are called *neighbors* if their Voronoi

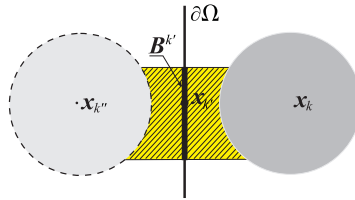


Fig. A.2. The quasidisk $B^{k'}$ centered at $\mathbf{x}_{k'}$ on the boundary of Ω

cells share a common edge.

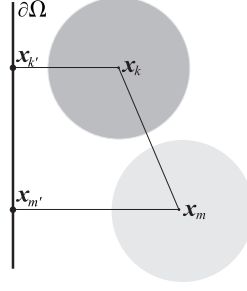


Fig. A.3. A “triangle” near the boundary of Ω

Now we can define a network \mathcal{G} as introduced in Subsection 2.

This graph is simply the Delaunay triangulation of X . Recall that the geometric dual of the Voronoi diagram in two dimensions is a Delaunay triangulation [5]. The basic element (or cell) of the Delaunay triangulation is a triangle that may degenerate into quadrilateral or n -gon, when four or more disk centers lie on the same circle.

The *neck-triangle partition* is a decomposition of the domain Ω_F into simple geometric objects: necks and triangles, based on the *central projection*.

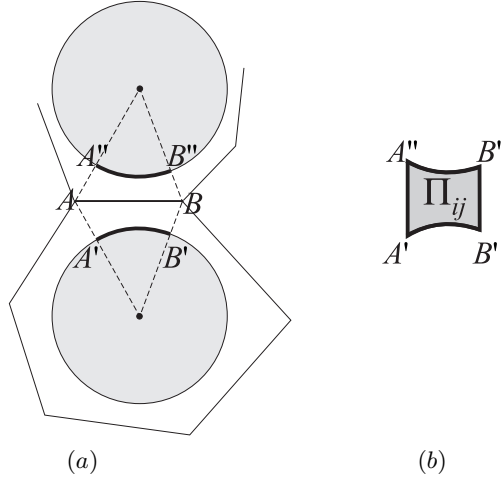


Fig. A.4. (a) Central projection projects the Voronoi edge AB onto arcs $A'B'$ and $A''B''$, (b) the neck Π_{ij}

The central projection partitions the boundary of a disk using the corresponding Voronoi cells as follows $\pi_i : \partial V(\mathbf{x}_i) \rightarrow \partial B^i$, $i \in \mathbb{I}$, where $\pi_i(\mathbf{x}) = R(\mathbf{x} - \mathbf{x}_i)/|\mathbf{x} - \mathbf{x}_i|$.

As shown on Fig. A.4, the *neck* Π_{ij} connects arcs $A'B'$ with the arc $A''B''$, obtained by the central projection of the same edge AB . We remark here that with this construction the edges of a neck are always parallel. Also a neck is necessarily symmetric with respect to the line connecting centers of the neighboring disks. The set of all necks in Ω_F is Π (the shadowed region on Fig. 2.4), called the fictitious fluid domain.

Once necks are defined the complementary part of the domain is the union of triangles (see e.g. \triangle_{ijk} on Fig. 2.4) and possibly trapezoids near the boundary $\partial\Omega$. As mentioned above, by slight abuse of terminology we call such trapezoids “triangles”. The set of all triangles is $\Delta = \bigcup_{i \in \mathbb{I}, j, k \in \mathcal{N}_i} \triangle_{ijk}$.

Thus, we obtain the *neck-triangle partition* of the fluid domain: $\Omega_F = \Delta \cup \Pi$. Such a partition of the fluid domain Ω_F allows us to decompose the functional W_{Ω_F} as follows:

$$\widehat{W} = W_{\Omega_F}(\cdot) = W_{\Pi}(\cdot) + W_{\Delta}(\cdot) =: \widehat{W}_{\Pi} + \widehat{W}_{\Delta}.$$

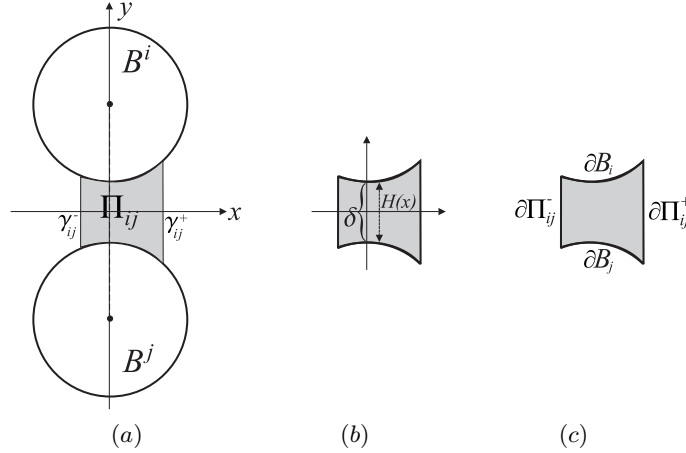


Fig. A.5. (a) The neck Π_{ij} connecting two neighbors B^i and B^j , (b) the distance $H(x)$ between two neighbors, (c) the boundary $\partial\Pi_{ij}$ of the neck

4 Auxiliary Problem for (3.1)

Here we prove Proposition 3.1. First we show that it is possible to choose such a density ρ_s and boundary data \mathbf{f} that disks do not move.

Theorem A.1. *There exist ρ_s and \mathbf{f} such that $\mathbf{U}^i = 0$ and $\omega^i = 0$, $i = 1, \dots, N$.*

Proof. On the periodicity cell $Y = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})$ (Fig.A.6) consider:

$$\left\{ \begin{array}{ll} (a) \mu \Delta \mathbf{v} = \nabla p, & \mathbf{x} \in Y \setminus B \\ (b) \nabla \cdot \mathbf{v} = 0, & \mathbf{x} \in Y \setminus B \\ (c) \mathbf{v} = \mathbf{0}, & \mathbf{x} \in \partial B \\ (d) \langle \mathbf{v} \rangle = \frac{1}{|Y|} \int_Y \mathbf{u} d\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ (e) \mathbf{v} \text{ is } Y\text{-periodic} \end{array} \right. \quad (\text{A.4})$$

This problem is equivalent to the one considered in [32].

The existence of the unique solution to (A.4) can be verified by the classical energy methods. Observe that the solution $\mathbf{v} = (v_1, v_2)$ of (A.4) satisfies the equations (a), (b) of the problem (3.1), and the condition (3.1c) (with $\mathbf{U}^i = \mathbf{0}$, $\omega^i = 0$) for $\Omega = Y$ (i.e. $N = 1$). Using symmetry of the problem one can show

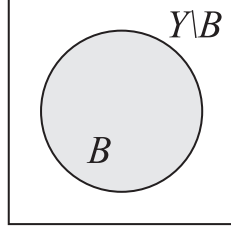


Fig. A.6. Periodicity cell Y

that \mathbf{v} also satisfies the balance of the angular momentum (3.1e) with $\Omega = Y$:

$$\int_{\partial B} \boldsymbol{\sigma}(\mathbf{v}) \mathbf{n} \times \mathbf{n} ds \left(= \int_{\partial B} ([\boldsymbol{\sigma}(\mathbf{v}) \mathbf{n}]_1 n_2 - [\boldsymbol{\sigma}(\mathbf{v}) \mathbf{n}]_2 n_1) ds \right) = 0,$$

$$\int_{\partial B} \boldsymbol{\sigma}(\mathbf{v}) \mathbf{n} ds = \begin{pmatrix} 0 \\ a \end{pmatrix},$$

for some $a \in \mathbb{R}$ which is not, in general, zero (this integral is the force exerted by the fluid on the inclusion B).

We estimate a in terms of δ . For this we compare problems (3.1) and (A.4) for \mathbf{u} and \mathbf{v} , respectively. If we choose \mathbf{f} in (3.1e) to be a restriction of \mathbf{v} on $\partial\Omega$ and $a = mg = \pi R^2(\rho_s - \rho_f)g$ then \mathbf{v} solves (3.1). Hence, $\mathbf{u} = \mathbf{v}$ in $\Omega = Y$, where, in particular, the translational and angular velocities are zeros: $\mathbf{U} = \mathbf{0}$, $\omega = 0$.

Consider now the problem (3.1) for \mathbf{u} obtained by a periodic extension of \mathbf{v} defined in Y to N cells. If we now choose ρ_s and \mathbf{f} as above, namely,

$$\rho_s = -\frac{a}{N\pi R^2 g} + \rho_f, \quad \mathbf{f} = \mathbf{v}|_{\partial\Omega}, \quad (\text{A.5})$$

then uniqueness implies that all inclusions are motionless (due to (A.4c)).
 \square

Theorem A.1 shows that if we choose the density ρ_s and the external boundary conditions \mathbf{f} in (3.1) so that inclusions are motionless then the only microflow between neighboring inclusions is permeation (or the Poiseuille microflow). We now show that the overall viscous dissipation rate of (3.1) exhibits a superstrong blow up. Hereinafter we use the equivalence of (3.1) and (A.4) extended by periodicity to Ω . Before we obtain the asymptotics of the dissipation rate \widehat{W} for this problem we find an asymptotics of ρ_s as $\delta \rightarrow 0$ and the form of \mathbf{f} .

Lemma A.2. *The following asymptotics holds for the force exerted by the fluid on the disks B defined by the right hand side of (A.4d):*

$$a = C\delta^{-5/2} + O(\delta^{-3/2}) \text{ with some constant } C = C(\mu, R, N) > 0 \text{ as } \delta \rightarrow 0.$$

Proof. Consider an auxiliary function $\mathbf{w} = \mathbf{v} - \langle \mathbf{v} \rangle$ that solves:

$$\left\{ \begin{array}{ll} (a) \mu \Delta \mathbf{w} = \nabla p, & \mathbf{x} \in Y \setminus B \\ (b) \nabla \cdot \mathbf{w} = 0, & \mathbf{x} \in Y \setminus B \\ (c) \mathbf{w} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, & \mathbf{x} \in \partial B \\ (d) \langle \mathbf{w} \rangle = \mathbf{0} \\ (e) \mathbf{w} \text{ is periodic} \end{array} \right. \quad (\text{A.6})$$

Then if we multiply (A.4a) by \mathbf{w} and integrate by parts over $Y \setminus B$ we obtain:

$$0 = - \int_{Y \setminus B} \mathbf{D}(\mathbf{w}) : \boldsymbol{\sigma}(\mathbf{v}) d\mathbf{x} + \int_{\partial B} \mathbf{w} \cdot \boldsymbol{\sigma}(\mathbf{v}) \mathbf{n} ds + \int_{\partial Y} \mathbf{w} \cdot \boldsymbol{\sigma}(\mathbf{v}) \mathbf{n} ds.$$

Taking into account that $\mathbf{D}(\mathbf{w}) = \mathbf{D}(\mathbf{v})$ and (A.6c) we continue

$$\begin{aligned} 0 &= - \int_{Y \setminus B} \mathbf{D}(\mathbf{v}) : \boldsymbol{\sigma}(\mathbf{v}) d\mathbf{x} + \int_{\partial B} [\boldsymbol{\sigma}(\mathbf{v}) \mathbf{n}]_2 ds + \int_{\partial Y} \mathbf{w} \cdot \boldsymbol{\sigma}(\mathbf{v}) \mathbf{n} ds \\ &= -W_{Y \setminus B}(\mathbf{v}) + a + \int_{\partial Y} \mathbf{w} \cdot \boldsymbol{\sigma}(\mathbf{v}) \mathbf{n} ds. \end{aligned}$$

Due to periodicity of the function \mathbf{w} , indicated above properties of the components of the vector $\boldsymbol{\sigma}(\mathbf{v}) \mathbf{n}$ and the symmetry of the domain Y we obtain that $\int_{\partial Y} \mathbf{w} \cdot \boldsymbol{\sigma}(\mathbf{v}) \mathbf{n} ds = 0$. Hence, $a = W_{Y \setminus B}(\mathbf{v})$. Therefore, in order to obtain an estimate for a with respect to δ we have to study a problem of the asymptotics of the overall viscous dissipation rate of the suspension given by (3.1). Since inclusions in problem (3.1) are motionless then the fluid permeates through thin gaps between them. So in view of formula (2.39) of Theorem 2.1

the viscous dissipation rate of this suspension exhibits the superstrong blow up of order $O(\delta^{-5/2})$ if we prove that $\beta_{ij} = O(1)$ for at least one pair (i, j) : $i \in \mathbb{I}, j \in \mathcal{N}_i$.

Recall that \mathbf{f} now is given by (A.5) and $v_1(-\frac{1}{2}, y) = v_1(\frac{1}{2}, y)$ due to periodicity of \mathbf{v} . But v_1 is odd in x , hence, v_1 does not depend on x . Similarly, $v_1(x, -\frac{1}{2}) = v_1(x, \frac{1}{2}) = 0$ due of oddness of v_1 in y . Thus, $v_1(x, y) = 0$. Moreover, due to the periodicity of $v_2(x, y)$ we have that the second component $f_2(x, y)$ of \mathbf{f} is even in both x and y . Hence, $\mathbf{f} = \begin{pmatrix} 0 \\ f_2(x, y) \end{pmatrix}$ where f_2 is an even function of both x and y on $\partial\Omega$.

Since for ρ_s and \mathbf{f} given by (A.5) we have $\mathbf{U}^i = \mathbf{0}$, and $\omega^i = 0$ for all i , the problem (3.1) becomes the problem of minimization of $W_{\Omega_F}(\mathbf{v})$ introduced in (2.3) over the class V defined by (2.4), where $\mathbf{U}^i = \mathbf{0}$, and $\omega^i = 0$ for all i , considered above. As mentioned above $a = \widehat{W}$ asymptotics of which is given by Propostion 2.38. To find an asymptotics of \widehat{W} we need to know the flux β_{ij} through the line ℓ_{ij} joining two neighboring disks B^i and B^j . In order to find this parameter consider a periodicity cell Y_F as in Fig. A.7(a). We note that due to divergence-free property and periodicity of \mathbf{u} one can obtain that β_{ij} is equal to the flux through any horizontal line ℓ_h connecting the lateral boundaries of Y_F :

$$\beta_{ij} = \frac{1}{R} \int_{\ell_{ij}} \mathbf{u} \cdot \mathbf{n} ds = \frac{1}{R} \int_{-\delta/2}^{\delta/2} u_2(x, 0) dx = \frac{1}{R} \int_{\ell_h} u_2(x, y) dx,$$

for any fixed $y \in (-1/2, 1/2)$. Using this fact, $\mathbf{u} = \mathbf{0}$ on ∂B , $\langle \mathbf{u} \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and

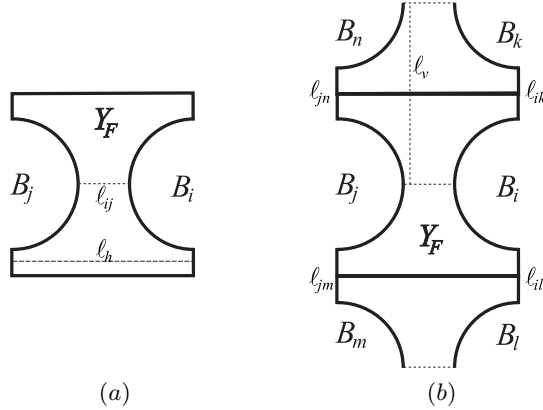


Fig. A.7. Periodicity cell Y_F

Fubini's theorem we have

$$1 = \frac{1}{|Y|} \int_Y u_2(x, y) d\mathbf{x} = \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \int_{-\frac{1}{2}}^{\frac{1}{2}} u_2(x, y) dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \int_{\ell_h} u_2(x, y) dx = R\beta_{ij}.$$

In Fig. A.7a ℓ_{ij} is shown as a horizontal line. There are also vertical half-middle lines connecting B^i and B^j with their neighbors in the vertical direction (B^k , B^l and B^n , B^m , respectively, in Fig. A.7b). As before using the periodicity and the divergence-free properties of \mathbf{u} we obtain that:

$$\beta_{ik} = \beta_{jn} = \frac{1}{R} \int_{\ell_v} u_1(x, y) dy \text{ for any fixed } x \in \left(-\frac{1}{2}, \frac{1}{2}\right).$$

where ℓ_v is any vertical line in Y_F (Fig. A.7b). And finally using the average of \mathbf{u} and that \mathbf{u} is zero on the inclusions we obtain:

$$0 = \frac{1}{|Y|} \int_Y u_1(x, y) d\mathbf{x} = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{\ell_v} u_1(x, y) dy = R\beta_{ik} = R\beta_{jn}.$$

Proceeding similarly one can obtain that $\beta_{jm} = \beta_{jn} = 0$.

Hence, we obtain that the flux through any horizontal line connecting the neighbors equals 1, while the flux through any vertical line is zero. Also we note that the number of pairs (i, j) for which β_{ij} is not zero is equal to the total number N of the inclusions in the suspension.

The fact that all inclusions are motionless and that there are N fluxes through the middle lines connecting neighbors of our network together with Proposition 2.1 results in $a = N\widehat{W} = \frac{9}{4}\pi\mu NR^{5/2}\delta^{-5/2} + O(\delta^{-3/2})$.

B Appendix

1 Details of construction of the trial fields

Verification of (6.18)

Consider the minimizer \mathbf{u}_{sh} of W over V_{sh} . The test function $\mathbf{v}_1 \in V_{sh}$ must satisfy the boundary conditions (6.3) on ∂B^i and ∂B^j , which can be rewritten as

$$\begin{aligned} \mathbf{G}_1^i &= (U_1^i - U_1^j + R\omega^i + R\omega^j) \left(\frac{1}{2} \mathbf{e}_1 \right) + R(\omega^i + \omega^j) \left(\frac{1}{2} \left[\sqrt{1 - \frac{x^2}{R^2}} - 1 \right] \right) \\ &=: (U_1^i - U_1^j + R\omega^i + R\omega^j) \mathbf{g}_1^i + R(\omega^i + \omega^j) \mathbf{g}_2^i, \\ \mathbf{G}_1^j &= (U_1^i - U_1^j + R\omega^i + R\omega^j) \left(-\frac{1}{2} \mathbf{e}_1 \right) + R(\omega^i + \omega^j) \left(-\frac{1}{2} \left[\sqrt{1 - \frac{x^2}{R^2}} - 1 \right] \right) \\ &=: (U_1^i - U_1^j + R\omega^i + R\omega^j) \mathbf{g}_1^j + R(\omega^i + \omega^j) \mathbf{g}_2^j. \end{aligned} \tag{B.1}$$

Hence, \mathbf{v}_1 can be written as

$$\mathbf{v}_1 = (U_1^i - U_1^j + R\omega^i + R\omega^j)\mathbf{v}_1^1 + R(\omega^i + \omega^j)\mathbf{v}_1^2, \quad (\text{B.2})$$

where

$$\mathbf{v}_1^1|_{\partial B^i} = \mathbf{g}_1^i, \quad \mathbf{v}_1^2|_{\partial B^i} = \mathbf{g}_2^i, \quad \mathbf{v}_1^1|_{\partial B^j} = \mathbf{g}_1^j, \quad \mathbf{v}_1^2|_{\partial B^j} = \mathbf{g}_2^j.$$

Using the decomposition (B.2) we define \mathbf{v}_1 by (6.21)-(6.22).

Next we evaluate the difference between $W(\mathbf{v}_1)$ and $W(\mathbf{u}_{sh})$. Here denote $\xi = U_1^i - U_1^j + R\omega^i + R\omega^j$ and $\zeta = R(\omega^i + \omega^j)$ then

$$W(\mathbf{u}_{sh}) = W(\xi \mathbf{u}_{sh}^1 + \zeta \mathbf{u}_{sh}^2),$$

where \mathbf{u}_{sh}^1 is the minimizer of W satisfying $\mathbf{g}_1^i, \mathbf{g}_1^j$ and \mathbf{u}_{sh}^2 is the minimizer of W satisfying $\mathbf{g}_2^i, \mathbf{g}_2^j$ on $\partial B^i, \partial B^j$, respectively.

Compute the functional $W(\mathbf{v}_1)$ as $\delta \rightarrow 0$:

$$\begin{aligned} W(\mathbf{v}_1) &= \xi^2 W(\mathbf{v}_1^1) + \zeta^2 W(\mathbf{v}_1^2) + 2\xi\zeta W(\mathbf{v}_1^1, \mathbf{v}_1^2) \\ &= \frac{1}{2}(U_1^i - U_1^j + R\omega^i + R\omega^j)^2 \pi \mu R^{1/2} \delta_{ij}^{-1/2} + \zeta^2 W(\mathbf{v}_1^2) + 2\xi\zeta W(\mathbf{v}_1^1, \mathbf{v}_1^2) \\ &=: (U_1^i - U_1^j + R\omega^i + R\omega^j)^2 \mathcal{C}_1^{ij} \delta^{-1/2} + \zeta^2 W(\mathbf{v}_1^2) + 2\xi\zeta W(\mathbf{v}_1^1, \mathbf{v}_1^2), \end{aligned} \quad (\text{B.3})$$

where $W(\mathbf{v}_1^2) = O(1)$, $W(\mathbf{v}_1^1, \mathbf{v}_1^2) = O(1)$ and

$$\mathcal{C}_1^{ij} = \frac{1}{2} \pi \mu R^{1/2} d_{ij}^{-1/2}. \quad (\text{B.4})$$

Due to parallelogram identity for any vector fields φ and ψ :

$$4W(\varphi, \psi) = W(\varphi + \psi) - W(\varphi - \psi),$$

we obtain

$$\begin{aligned} |W(\mathbf{v}_1) - W(\mathbf{u}_{sh})| &= |W(\xi \mathbf{v}_1^1 + \zeta \mathbf{v}_1^2) - W(\xi \mathbf{u}_{sh}^1 + \zeta \mathbf{u}_{sh}^2)| \\ &\leq \xi^2 |W(\mathbf{v}_1^1) - W(\mathbf{u}_{sh}^1)| + \zeta^2 |W(\mathbf{v}_1^2) - W(\mathbf{u}_{sh}^2)| \\ &\quad + \frac{\xi\zeta}{2} |W(\mathbf{v}_1^1 + \mathbf{v}_1^2) - W(\mathbf{u}_{sh}^1 + \mathbf{u}_{sh}^2)| + \frac{\xi\zeta}{2} |W(\mathbf{v}_1^1 - \mathbf{v}_1^2) - W(\mathbf{u}_{sh}^1 - \mathbf{u}_{sh}^2)|. \end{aligned} \quad (\text{B.5})$$

In order to evaluate these differences in (B.5) we construct four trial tensors $\mathcal{S}_1^k \in F$, $k = 1, \dots, 4$, such that,

$$\begin{aligned} W_1^*(\mathcal{S}_1^1) &\leq W_1^*(\sigma(\mathbf{u}_{sh}^1)), & W_1^*(\mathcal{S}_1^2) &\leq W_1^*(\sigma(\mathbf{u}_{sh}^2)), \\ W_1^*(\mathcal{S}_1^3) &\leq W_1^*(\sigma(\mathbf{u}_{sh}^1 + \mathbf{u}_{sh}^2)), & W_1^*(\mathcal{S}_1^4) &\leq W_1^*(\sigma(\mathbf{u}_{sh}^1 - \mathbf{u}_{sh}^2)), \end{aligned}$$

obtaining

$$|W(\mathbf{v}_1) - W(\mathbf{u}_{sh})| \leq \xi^2 |W(\mathbf{v}_1^1) - W_1^*(\mathcal{S}_1^1)| + \zeta^2 [W(\mathbf{v}_1^2) - W_1^*(\mathcal{S}_1^4)] \\ + \frac{\xi\zeta}{2} |W(\mathbf{v}_1^1 + \mathbf{v}_1^2) - W_1^*(\mathcal{S}_1^2)| + \frac{\xi\zeta}{2} |W(\mathbf{v}_1^1 - \mathbf{v}_1^2) - W_1^*(\mathcal{S}_1^3)|.$$

To construct e.g. \mathcal{S}_1^1 we use the following observation. If \mathbf{v}_1^1 is a “good” approximation of \mathbf{u}_{sh}^1 then we expect that $\boldsymbol{\sigma}(\mathbf{v}_1^1) = 2\mu\mathbf{D}(\mathbf{v}_1^1) - p_{v_1^1}\mathbf{I}$, where

$$\mathbf{D}(\mathbf{v}_1^1) = \begin{pmatrix} yG' & \frac{1}{2}(G + F' - \frac{y^2}{2}G'') \\ \frac{1}{2}(G + F' - \frac{y^2}{2}G'') & -yG' \end{pmatrix}, \quad p_{v_1^1} = 0,$$

with $G(x)$ and $F(x)$ defined in (6.22), is a “good” approximation of $\boldsymbol{\sigma}(\mathbf{u}_{sh}^1)$. Then we correct $\boldsymbol{\sigma}(\mathbf{v}_1^1)$ so that it belongs to the set F , thus, obtaining \mathcal{S}_1^1 .

Now define

$$\mathcal{S}_1^1 = \mu \begin{pmatrix} 0 & G(x) - C \\ G(x) - C & -yG'(x) \end{pmatrix} \quad \text{in } \Pi_{ij}, \quad (\text{B.6})$$

with $G(x)$ given by (6.22), and the constant $C = \frac{1}{H(\gamma_{ij})}$. Clearly, $\mathcal{S}_1^1 \in F$.

When we evaluate the difference between $W(\mathbf{v}_1^1)$ defined by (B.3) and $W_1^*(\mathcal{S}_1^1)$, defined by (6.13) we obtain the following estimate:

$$|W(\mathbf{v}_1^1) - W_1^*(\mathcal{S}_1^1)| = O(1), \quad \text{as } \delta \rightarrow 0.$$

Now choose

$$\mathcal{S}_1^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in F,$$

hence, $W_1^*(\mathcal{S}_1^2) = 0$.

We also choose $\mathcal{S}_1^3 = \mathcal{S}_1^1$ and $\mathcal{S}_1^4 = \mathcal{S}_1^1$ and compute the corresponding functionals obtaining

$$|W(\mathbf{v}_1^1 + \mathbf{v}_1^2) - W_1^*(\mathcal{S}_1^1)| = O(1), \quad |W(\mathbf{v}_1^1 - \mathbf{v}_1^2) - W_1^*(\mathcal{S}_1^1)| = O(1),$$

as $\delta \rightarrow 0$. Therefore, $W(\mathbf{v}_1)$ determines the asymptotics of $W(\mathbf{u}_{sh})$:

$$|W(\mathbf{v}_1) - W(\mathbf{u}_{sh})| \leq E_1(\mathbf{U}^i, \mathbf{U}^j, \omega^i, \omega^j), \quad \text{as } \delta \rightarrow 0, \quad (\text{B.7})$$

where the error term E_1 is defined in (6.18). Hence,

$$W(\mathbf{u}_{sh}) = [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij} + R\omega^i + R\omega^j]^2 \mathcal{C}_1^{ij} \delta^{-1/2} + E_1(\mathbf{U}^i, \mathbf{U}^j, \omega^i, \omega^j) \quad \text{as } \delta \rightarrow 0. \quad (\text{B.8})$$

Verification of (6.19)

Consider the minimizer \mathbf{u}_{sq} of W over V_{sq} . Choose \mathbf{v}_2 by (6.24)-(6.25). Introduce an approximate pressure

$$p_{v_2} = \mu \left(6 \int_{\gamma_0}^x F(t) dt + G' - 3y^2 F' \right), \quad (\text{B.9})$$

where γ_0 is some constant and compute

$$\sigma(v_2) = (U_2^i - U_2^j) \begin{pmatrix} 2\mu(G' + 3y^2 F') - p_{v_2} & \mu(6yF - yG'' - y^3 F'') \\ \mu(6yF - yG'' - y^3 F'') & -2\mu(G' + 3y^2 F') - p_{v_2} \end{pmatrix}. \quad (\text{B.10})$$

Now we construct a corrector $\mathcal{S}_2 = (\mathcal{S}_2)_{i,j=1,2}$ to (B.10) so that $\mathcal{S}_2 \in F$. Define \mathcal{S}_2 by (6.26) where constants C_i ($i = 1..4$) are chosen so that $\mathcal{S}_2 \mathbf{n} = \mathbf{0}$ on $\partial \Pi_{ij}^\pm$, that is,

$$\begin{aligned} C_1 &= \frac{G''(\gamma_{ij})}{\gamma_{ij}} - 6 \frac{F(\gamma_{ij})}{\gamma_{ij}}, & C_2 &= 3 \frac{F''(\gamma_{ij})}{\gamma_{ij}}, \\ C_3 &= -3G'(\gamma_{ij}) + C_1 \frac{\gamma_{ij}^2}{2}, & C_4 &= -9F'(\gamma_{ij}) + \frac{3}{2} C_2 \gamma_{ij}^2. \end{aligned}$$

Then we compute the functional $W(v_2)$ and obtain

$$W(v_2) = (U_2^i - U_2^j)^2 \left(\mathcal{C}_2^{ij} \delta^{-3/2} + \mathcal{C}_3^{ij} \delta^{-1/2} \right) + C_4 \mu (U_2^i - U_2^j)^2, \quad (\text{B.11})$$

where

$$\mathcal{C}_2^{ij} = \frac{3}{8} \pi \mu R^{3/2} d_{ij}^{-3/2}, \quad \mathcal{C}_3^{ij} = \frac{207}{320} \pi \mu R^{1/2} d_{ij}^{-1/2}. \quad (\text{B.12})$$

When we evaluate the difference between $W(v_2)$, defined by (B.11), and $W_2^*(\mathcal{S}_2)$, defined by (6.13) we obtain the following estimate:

$$|W(v_2) - W_2^*(\mathcal{S}_2)| \leq E_2(U^i, U^j), \quad \text{as } \delta \rightarrow 0, \quad (\text{B.13})$$

where E_2 is defined by (6.19). Therefore, $W(v_2)$ determines the asymptotics of $W(u_{sq})$ as $\delta \rightarrow 0$:

$$W(u_{sq}) = ([U^i - U^j] \cdot \mathbf{q}^{ij})^2 \left(\mathcal{C}_2^{ij} \delta^{-3/2} + \mathcal{C}_3^{ij} \delta^{-1/2} \right) + E_2(U^i, U^j). \quad (\text{B.14})$$

Verification of (6.20)

Consider the minimizer u_{per} of W over V_{per} . The test function $v_3 \in V_{per}$ may be written in the form:

$$v_3 = R(\omega^i - \omega^j) v_3^1 + \beta_{ij} R v_3^2, \quad (\text{B.15})$$

where

$$v_3^1|_{\partial B^i} = \mathbf{G}_3^i, \quad v_3^1|_{\partial B^j} = \mathbf{G}_3^j, \quad v_3^2|_{\partial B^i} = \mathbf{0}, \quad v_3^2|_{\partial B^j} = \mathbf{0}.$$

and the parameter β_{ij} is defined as follows. The flux of v_3 through ℓ_{ij} should satisfy $\int_{\ell_{ij}} v_3 \cdot \mathbf{n} ds = \beta_{ij}^* - \frac{\delta_{ij}}{2R} (U_1^i + U_1^j)$, then

$$\beta_{ij} = \beta_{ij}^* - \frac{\omega^i - \omega^j}{R} \int_{\ell_{ij}} \mathbf{v}_3^1 \cdot \mathbf{n} ds - \frac{\delta_{ij}}{2R} (U_1^i + U_1^j), \quad (\text{B.16})$$

where $\frac{\delta_{ij}}{2R} (U_1^i + U_1^j) = \int_{\ell_{ij}} \mathbf{u}_t \cdot \mathbf{n} ds$.

Define \mathbf{v}_3 by (6.27)-(6.28). Computing $\int_{\ell_{ij}} \mathbf{v}_3^1 \cdot \mathbf{n} ds = \frac{\delta_{ij}}{2} + \frac{\delta_{ij}^2}{8R}$ we obtain that β_{ij} is defined by (B.16).

Now we rewrite (6.27) as one vector:

$$\mathbf{v}_3 = \begin{pmatrix} G(x) + 3y^2 F(x) \\ -yG'(x) - y^3 F'(x) \end{pmatrix} \quad \text{in } \Pi_{ij}, \quad (\text{B.17})$$

where

$$G(x) = R(\omega^i - \omega^j)P(x) + \beta_{ij}RK(x), \quad F(x) = R(\omega^i - \omega^j)Q(x) + \beta_{ij}RM(x), \quad (\text{B.18})$$

with P, Q, K and M defined by (6.28).

As before to choose a trial tensor $\mathcal{S}_3 \in F_{per}$ we look at the stress tensor $\boldsymbol{\sigma}(\mathbf{v}_3) = 2\mu\mathbf{D}(\mathbf{v}_3) - p_{v_3}$, where an approximate pressure p_{v_3} can be chosen as in (B.9) for G, F given by (B.18).

Then for such functions G and F define the trial tensor $\mathcal{S}_3 \in F_{per}$ by (6.29) where constants C_i as \mathcal{S}_2 .

Next we compute the functional $W(\mathbf{v}_3)$:

$$\begin{aligned} W(\mathbf{v}_3) &= \beta_{ij}^2 \left(\mathcal{C}_4^{ij} \delta^{-5/2} + \mathcal{C}_5^{ij} \delta^{-3/2} + \mathcal{C}_6^{ij} \delta^{-1/2} \right) \\ &\quad + R(\omega^i - \omega^j) \beta_{ij} \left(\mathcal{C}_7^{ij} \delta^{-3/2} + \mathcal{C}_8^{ij} \delta^{-1/2} \right) \\ &\quad + R^2 (\omega^i - \omega^j)^2 \mathcal{C}_9^{ij} \delta^{-1/2} \\ &\quad + \mathcal{C}_5 R^2 (\omega^i - \omega^j)^2 + \mathcal{C}_6 R^2 (\omega^i - \omega^j) \beta_{ij} + \mathcal{C}_7 R^2 \beta_{ij}^2, \end{aligned} \quad (\text{B.19})$$

where

$$\begin{aligned} \mathcal{C}_4^{ij} &= \frac{9}{4} \pi \mu R^{5/2} d_{ij}^{-5/2}, \quad \mathcal{C}_5^{ij} = \frac{99}{160} \pi \mu R^{3/2} d_{ij}^{-3/2}, \quad \mathcal{C}_6^{ij} = \frac{29241}{17920} \pi \mu R^{1/2} d_{ij}^{-1/2} \\ \mathcal{C}_7^{ij} &= -3 \pi \mu R^{3/2} d_{ij}^{-3/2}, \quad \mathcal{C}_8^{ij} = \frac{9}{40} \pi \mu R^{1/2} d_{ij}^{-1/2}, \quad \mathcal{C}_9^{ij} = \frac{3}{2} \pi \mu R^{1/2} d_{ij}^{-1/2}. \end{aligned} \quad (\text{B.20})$$

When we evaluate the difference between $W(\mathbf{v}_3)$, defined by (B.19)-(B.20), and $W_3^*(\mathcal{S}_3)$, defined by (6.14) we obtain the following estimate:

$$|W(\mathbf{v}_3) - W_3^*(\mathcal{S}_3)| \leq E_3(\omega^i, \omega^j, \beta_{ij}), \quad \text{as } \delta \rightarrow 0, \quad (\text{B.21})$$

where E_3 is defined by (6.20). Therefore, $W(\mathbf{v}_3)$ determines the asymptotics of $W(\mathbf{u}_{per})$:

$$\begin{aligned}
W(\mathbf{u}_{per}) &= \beta_{ij}^2 \left(\mathcal{C}_4^{ij} \delta^{-5/2} + \mathcal{C}_5^{ij} \delta^{-3/2} + \mathcal{C}_6^{ij} \delta^{-1/2} \right) \\
&\quad + R(\omega^i - \omega^j) \beta_{ij} \left(\mathcal{C}_7^{ij} \delta^{-3/2} + \mathcal{C}_8^{ij} \delta^{-1/2} \right) \\
&\quad + R^2(\omega^i - \omega^j)^2 \mathcal{C}_9^{ij} \delta^{-1/2} + E_3(\omega^i, \omega^j, \beta_{ij}).
\end{aligned} \tag{B.22}$$

2 Construction trial fields in boundary neck Π_{ij} , ($i \in \mathbb{I}$, $j \in \mathcal{N}_i \cap \mathbb{B}$)

The construction of the trial vector and tensor fields in neck Π_{ij} connecting the disk B^i and the quasidisk B^j as in Fig. B.1 can be done similarly to the case of the neck connecting two interior disks. Let B^i be a disk near the boundary and B^j a quasidisk, neighboring to B_i . We call \mathbf{U}^i, ω^i of the quasidisk B^j the “translational” and the “angular” velocities of the quasidisk B^j , respectively. Now consider the neck Π_{ij} connecting the disk B^i and the quasidisk B^j lying on the lower part of $\partial\Omega$. The other cases are treated similarly.

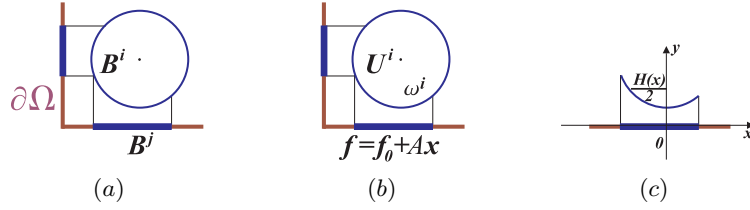


Fig. B.1. (a) The disk B^i and quasidisk B^j , (b) boundary conditions on B^i and B^j , (c) the local coordinate system

Recall that in general necks are not symmetric. To construct a test function $\mathbf{v} \in V_{ij}$ in the neck Π_{ij} connecting the disk B^i and quasidisk B^j (Fig. B.1(a)) we extend Π_{ij} to

$$\widehat{\Pi}_{ij} = \{(x, y) \in \mathbb{R}^2 : -\gamma_{ij} < x < \gamma_{ij}, -H_{ij}(x)/2 < y < H_{ij}(x)/2\}, \tag{B.23}$$

where

$$\gamma_{ij} = \begin{cases} \gamma_{ij}^-, & \text{if } \gamma_{ij}^- > \gamma_{ij}^+, \\ \gamma_{ij}^+, & \text{if } \gamma_{ij}^+ > \gamma_{ij}^-, \end{cases}$$

and construct $\widehat{\mathbf{v}} \in H^1(\widehat{\Pi}_{ij})$ such that $\widehat{\mathbf{v}} = \mathbf{U}^i + R\omega^i(n_1\mathbf{e}_2 - n_2\mathbf{e}_1)$ on $\partial B^i \cap \partial\widehat{\Pi}_{ij}$, $\widehat{\mathbf{v}} = A\mathbf{x}$ on $\partial B^j \cap \partial\widehat{\Pi}_{ij}$ and $\frac{1}{R} \int_{\ell_{ij}} \widehat{\mathbf{v}} \cdot \mathbf{n} ds = \beta_{ij}^*$. Then the restriction of this function on Π_{ij} : $\mathbf{v} = \widehat{\mathbf{v}}|_{\Pi_{ij}}$ would be a function from V_{ij} (since this would be $H^1(\Pi_{ij})$ function and all boundary conditions would be satisfied). Hence, $W_{\Pi_{ij}}(\mathbf{u}) \leq W_{\Pi_{ij}}(\mathbf{v})$, where \mathbf{u} is the actual minimizer of $W_{\Pi_{ij}}$ in V_{ij} .

We can decompose a test function $\mathbf{v} \in V_{ij}$ ($i \in \mathbb{I}, j \in \mathcal{N}_i \cap \mathbb{B}$) in to the sum $\mathbf{v} = \mathbf{u}_t + \mathbf{v}_1 + \mathbf{v}_2$ so that $W_{\Pi_{ij}}(\mathbf{v}) = W_{\Pi_{ij}}(\mathbf{v}_1) + W_{\Pi_{ij}}(\mathbf{v}_2)$, and $W_{\Pi_{ij}}(\mathbf{v}_3) = 0$ where functions $\mathbf{v}_1, \mathbf{v}_2$ satisfy

$$\begin{aligned}
(a) \quad & \mathbf{u}_t = \frac{1}{2}(\mathbf{U}^i + \mathbf{U}^j) \text{ for } \mathbf{x} \in \partial B^i \text{ and } \mathbf{x} \in \partial B^j, \\
(b) \quad & \mathbf{v}_1 = \begin{pmatrix} U_1^i \\ 0 \end{pmatrix} + \omega^i \begin{pmatrix} \sqrt{R^2 - x^2} \\ x \end{pmatrix} \text{ for } \mathbf{x} \in \partial B^i, \mathbf{v}_1 = \begin{pmatrix} U_1^j \\ \omega^j x \end{pmatrix} \text{ for } \mathbf{x} \in \partial B^j, \\
& \text{with } \int_{\ell_{ij}} \mathbf{v}_1 \cdot \mathbf{n} ds = \beta_{ij}^* R - \int_{\ell_{ij}} \mathbf{u}_t \cdot \mathbf{n} ds = \beta_{ij}^* R - \frac{1}{4}[(\mathbf{U}^i + \mathbf{U}^j) \cdot \mathbf{p}^{ij}] \delta_{ij}, \\
(c) \quad & \mathbf{v}_2 = \begin{pmatrix} 0 \\ U_2^i \end{pmatrix} \text{ for } \mathbf{x} \in \partial B^i, \mathbf{v}_2 = \begin{pmatrix} ax \\ U_2^j \end{pmatrix} \text{ for } \mathbf{x} \in \partial B^j.
\end{aligned} \tag{B.24}$$

Then the corresponding decomposition of the minimizer is: $\mathbf{u} = \mathbf{u}_t + \mathbf{u}_1 + \mathbf{u}_2$.

In the local coordinate system we choose these functions $\mathbf{v}_1, \mathbf{v}_2$ to be as follows:

$$\mathbf{v}_1 = \begin{pmatrix} U_1^j + yK(x) + y^2M(x) \\ \omega^j x - \frac{y^2}{2}K'(x) - \frac{y^3}{3}M'(x) \end{pmatrix} \quad \text{in } \Pi_{ij}, \tag{B.25}$$

where

$$\begin{aligned}
K(x) &= -\frac{12(\omega^i - \omega^j)x^2}{H(x)^2} + \frac{8(U_1^i - U_1^j + R\omega^i)}{H(x)} - \frac{8\omega^i}{H(x)}(R - \sqrt{R^2 - x^2}) + \frac{24R\beta_{ij}}{H(x)^2}, \\
M(x) &= \frac{24(\omega^i - \omega^j)x^2}{H(x)^3} - \frac{12(U_1^i - U_1^j + R\omega^i)}{H(x)^2} + \frac{12\omega^i}{H(x)^2}(R - \sqrt{R^2 - x^2}) - \frac{48R\beta_{ij}}{H(x)^3},
\end{aligned}$$

with

$$\beta_{ij} = \beta_{ij}^* - \frac{3\delta_{ij}}{4R}U_1^i - \frac{\delta_{ij}}{4R}U_1^j - \frac{\delta_{ij}}{2}\omega^i = \beta_{ij}^* - \frac{3\delta_{ij}}{4R}\mathbf{U}^i \cdot \mathbf{p}^{ij} - \frac{\delta_{ij}}{4R}\mathbf{U}^j \cdot \mathbf{p}^{ij} - \frac{\delta_{ij}}{2}\omega^i,$$

and

$$\mathbf{v}_2 = \begin{pmatrix} ax + yF(x) + y^2G(x) \\ U_2^j - ay - \frac{y^2}{2}F'(x) - \frac{y^3}{3}G'(x) \end{pmatrix} \quad \text{in } \Pi_{ij}, \tag{B.26}$$

where

$$F(x) = -\frac{8ax}{H(x)} - \frac{24(U_2^i - U_2^j)x}{H(x)^2}, \quad G(x) = \frac{12ax}{H(x)^2} + \frac{48(U_2^i - U_2^j)x}{H(x)^3}.$$

We construct a lower bound for the viscous dissipation rate in the neck near the boundary we consider a dual variational problem in a symmetric neck $\widehat{\Pi}_{ij}$ (B.23) with

$$\gamma_{ij} = \begin{cases} \gamma_{ij}^-, & \text{if } \gamma_{ij}^- < \gamma_{ij}^+, \\ \gamma_{ij}^+, & \text{if } \gamma_{ij}^+ < \gamma_{ij}^-, \end{cases}$$

that is, the problem of the maximization of the dual functional (6.13) with \mathbf{G}_3^i and \mathbf{G}_3^j given by (B.24a) in the dual space F_{per} given by (6.16), all defined over the neck $\widehat{\Pi}_{ij}$. Indeed,

$$W_{\Pi_{ij}}(\mathbf{u}) \geq W_{\widehat{\Pi}_{ij}}(\mathbf{u}) \geq W_{\widehat{\Pi}_{ij}}^*(\mathcal{S}).$$

Here the test tensor \mathcal{S} is decomposed into the sum $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$, with $\mathcal{S}_1 \in F_{per}$, $\mathcal{S}_2 \in F$ corresponding to two trial vector fields $\mathbf{v}_1, \mathbf{v}_2$ defined above.

We choose

$$\mathcal{S}_1 = \begin{pmatrix} -2 \int_{\gamma_0}^x M(t)dt + C_1 x & K + 2yM - C_1 y - C_2 \\ K + 2yM - C_1 y - C_2 & -yK' - y^2 M' \end{pmatrix} \quad \text{in } \widehat{\Pi}_{ij}, \quad (\text{B.27})$$

where constants $C_1 = 2M(\gamma_{ij})$, $C_2 = K(\gamma_{ij})$. This tensor is divergence free in $\widehat{\Pi}_{ij}$, and on the lateral boundary $\partial \widehat{\Pi}_{ij}^\pm$ of the neck $\widehat{\Pi}_{ij}$ it takes values:

$$\chi^+ = -2 \int_{\gamma_0}^{\gamma_{ij}} M(x)dx + C_1 \gamma_{ij}, \quad \chi^- = -2 \int_{\gamma_0}^{-\gamma_{ij}} M(t)dt - C_1 \gamma_{ij},$$

hence $\mathcal{S}_1 \in F_{per}$.

Now we compute the functional $W(\mathbf{v}_1)$ in the neck connecting a disk and a quasidisk:

$$\begin{aligned} W(\mathbf{v}_1) &= \beta_{ij}^2 \left[18\pi\mu R^{5/2} \delta_{ij}^{-5/2} + \frac{51}{20} \pi\mu R^{3/2} \delta_{ij}^{-3/2} + \frac{20889}{2240} \pi\mu R^{1/2} \delta_{ij}^{-1/2} \right] \\ &\quad + 6\pi\mu R^{3/2} \beta_{ij} \left(U_1^i - U_1^j + R\omega^i \right) \delta_{ij}^{-3/2} - 3\pi\mu R^{3/2} \beta_{ij} R(\omega^i - \omega^j) \delta_{ij}^{-3/2} \\ &\quad + \frac{19}{20} \pi\mu R^{1/2} \beta_{ij} \left(U_1^i - U_1^j + R\omega^i \right) \delta_{ij}^{-1/2} - \frac{3}{8} \pi\mu R^{1/2} \beta_{ij} R(\omega^i - \omega^j) \delta_{ij}^{-1/2} \\ &\quad - 3\pi\mu R^{1/2} \beta_{ij} R\omega^i \delta_{ij}^{-1/2} + 4\pi\mu R^{1/2} \left(U_1^i - U_1^j + R\omega^i \right)^2 \delta_{ij}^{-1/2} \\ &\quad - 3\pi\mu R^{1/2} \left(U_1^i - U_1^j + R\omega^i \right) R(\omega^i - \omega^j) \delta_{ij}^{-1/2} \\ &\quad + \frac{9}{2} \pi\mu R^{1/2} R^2 (\omega^i - \omega^j)^2 \delta_{ij}^{-1/2} + E_1^b(\mathbf{U}^i, \omega^i, \mathbf{f}, \beta_{ij}) \\ &=: \beta_{ij}^2 \left[\mathcal{C}_1^{ij} \delta_{ij}^{-5/2} + \mathcal{C}_2^{ij} \delta_{ij}^{-3/2} + \mathcal{C}_3^{ij} \delta_{ij}^{-1/2} \right] \\ &\quad + \mathcal{C}_4^{ij} \beta_{ij} \left(U_1^i - U_1^j + R\omega^i \right) \delta_{ij}^{-3/2} + \mathcal{C}_5^{ij} \beta_{ij} R(\omega^i - \omega^j) \delta_{ij}^{-3/2} \\ &\quad + \mathcal{C}_6^{ij} \beta_{ij} \left(U_1^i - U_1^j + R\omega^i \right) \delta_{ij}^{-1/2} + \mathcal{C}_7^{ij} \beta_{ij} R(\omega^i - \omega^j) \delta_{ij}^{-1/2} \\ &\quad + \mathcal{C}_8^{ij} \beta_{ij} R\omega^i \delta_{ij}^{-1/2} + \mathcal{C}_9^{ij} \left(U_1^i - U_1^j + R\omega^i \right)^2 \delta_{ij}^{-1/2} \\ &\quad + \mathcal{C}_{10}^{ij} \left(U_1^i - U_1^j + R\omega^i \right) R(\omega^i - \omega^j) \delta_{ij}^{-1/2} \\ &\quad + \mathcal{C}_{11}^{ij} R^2 (\omega^i - \omega^j)^2 \delta_{ij}^{-1/2} + E_1^b(\mathbf{U}^i, \omega^i, \mathbf{f}, \beta_{ij}), \end{aligned} \quad (\text{B.28})$$

where the error term

$$E_1^b(\mathbf{U}^i, \omega^i, \mathbf{f}, \beta_{ij}) = \mu \left(C_7 \beta_{ij}^2 + C_8 [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij}]^2 + C_9 R^2 (\omega^i)^2 + C_{10} R^2 (\omega^j)^2 \right) \quad (\text{B.29})$$

with universal constants C_k , $k = 7, \dots, 10$ and

$$\begin{aligned}
\mathbf{c}_1^{ij} &= 18\pi\mu R^{5/2}d_{ij}^{-5/2}, \quad \mathbf{c}_2^{ij} = \frac{51}{20}\pi\mu R^{3/2}d_{ij}^{-3/2}, \quad \mathbf{c}_3^{ij} = \frac{20889}{2240}\pi\mu R^{1/2}d_{ij}^{-1/2}, \\
\mathbf{c}_4^{ij} &= 6\pi\mu R^{3/2}d_{ij}^{-3/2}, \quad \mathbf{c}_5^{ij} = -3\pi\mu R^{3/2}d_{ij}^{-3/2}, \quad \mathbf{c}_6^{ij} = \frac{19}{20}\pi\mu R^{1/2}d_{ij}^{-1/2}, \\
\mathbf{c}_7^{ij} &= -\frac{3}{8}\pi\mu R^{1/2}d_{ij}^{-1/2}, \quad \mathbf{c}_8^{ij} = -3\pi\mu R^{1/2}\delta_{ij}^{-1/2}, \quad \mathbf{c}_9^{ij} = 4\pi\mu R^{1/2}d_{ij}^{-1/2}, \\
\mathbf{c}_{10}^{ij} &= -3\pi\mu R^{1/2}d_{ij}^{-1/2}, \quad \mathbf{c}_{11}^{ij} = \frac{9}{2}\pi\mu R^{1/2}d_{ij}^{-1/2}.
\end{aligned}$$

When we evaluate the difference between $W(\mathbf{v}_1)$, defined by (B.28), and $W_3^*(\mathcal{S}_1)$, defined by (6.14) with the functions $\mathbf{G}_k^i, \mathbf{G}_k^j$ given by (B.24a) we obtain the following estimate:

$$|W(\mathbf{v}_1) - W_2^*(\mathcal{S}_1)| \leq E_1^b(\mathbf{U}^i, \omega^i, \mathbf{f}, \beta_{ij}), \quad \text{as } \delta \rightarrow 0. \quad (\text{B.30})$$

Therefore, $W(\mathbf{v}_1)$ determines the asymptotics of $W(\mathbf{u}_1)$ in the neck connecting the disk and quasidisk:

$$\begin{aligned}
W(\mathbf{u}_1) &= \beta_{ij}^2 \left[\mathbf{B}_1^{ij} \delta^{-5/2} + \mathbf{B}_2^{ij} \delta^{-3/2} + \mathbf{B}_3^{ij} \delta^{-1/2} \right] \\
&+ \mathbf{B}_4^{ij} \beta_{ij} [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij} + R\omega^i] \delta^{-3/2} + \mathbf{B}_5^{ij} \beta_{ij} R(\omega^i - \omega^j) \delta^{-3/2} \\
&+ \mathbf{B}_6^{ij} \beta_{ij} [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij} + R\omega^i] \delta^{-1/2} + \mathbf{B}_7^{ij} \beta_{ij} R(\omega^i - \omega^j) \delta^{-1/2} \\
&+ \mathbf{B}_8^{ij} \beta_{ij} R\omega^i \delta^{-1/2} + \mathbf{B}_9^{ij} [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij} + R\omega^i]^2 \delta^{-1/2} \\
&+ \mathbf{B}_{10}^{ij} [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij} + R\omega^i] R(\omega^i - \omega^j) \delta^{-1/2} \\
&+ \mathbf{B}_{11}^{ij} R^2(\omega^i - \omega^j)^2 \delta^{-1/2} + E_1^b(\mathbf{U}^i, \omega^i, \mathbf{f}, \beta_{ij}).
\end{aligned} \quad (\text{B.31})$$

Now we choose $\mathcal{S}_2 \in F$ as follows:

$$\begin{aligned}
(\mathcal{S}_2)_{11} &= \mu \left(3yF' + 3y^2G' - 2 \int_{\gamma_0}^x G + \frac{x^2}{2} (C_2 + C_3y + C_4y^2) \right. \\
&\quad \left. - (C_5 + C_6y + C_7y^2) \right), \\
(\mathcal{S}_2)_{12} &= (\mathcal{S}_2)_{21} = \mu \left(F + 2yG - \frac{3}{2}y^2F'' - y^3G'' - C_1x - C_2xy - C_3x\frac{y^2}{2} - C_4x\frac{y^3}{3} \right), \\
(\mathcal{S}_2)_{22} &= \mu \left(-yF' - y^2G' - 2 \int_{\gamma_0}^x G + \frac{1}{2}y^3F''' + \frac{1}{4}y^4G''' \right. \\
&\quad \left. + C_1y + C_2\frac{y^2}{2} + C_3\frac{y^3}{6} + C_4\frac{y^4}{12} \right),
\end{aligned} \quad (\text{B.32})$$

where constants C_i ($i = 1..7$) are chosen so that $\mathcal{S}_2 \mathbf{n} = \mathbf{0}$ on $\partial \Pi_{ij}^\pm$, that is,

$$\begin{aligned}
C_1 &= \frac{1}{\gamma_{ij}} F(\gamma_{ij}), \quad C_2 = \frac{2}{\gamma_{ij}} G(\gamma_{ij}), \quad C_3 = -\frac{3}{\gamma_{ij}} F''(\gamma_{ij}), \quad C_4 = -\frac{3}{\gamma_{ij}} G''(\gamma_{ij}), \\
C_5 &= C_2 \frac{\gamma_{ij}^2}{2} - 2 \int_{\gamma_0}^{\gamma_{ij}} G, \quad C_6 = C_3 \frac{\gamma_{ij}^2}{2} + 3F'(\gamma_{ij}), \quad C_7 = C_4 \frac{\gamma_{ij}^2}{2} + 3G'(\gamma_{ij}).
\end{aligned}$$

We now compute the functional $W(\mathbf{v}_2)$:

$$\begin{aligned}
W(\mathbf{v}_2) &= (U_2^i - U_2^j)^2 \left[6\pi\mu R^{3/2} \delta_{ij}^{-3/2} + \frac{63}{20} \pi\mu R^{1/2} \delta_{ij}^{-1/2} \right] \\
&\quad + 6\pi\mu R^{1/2} (U_2^i - U_2^j) Ra \delta_{ij}^{-1/2} + E_2^b(\mathbf{U}^i, \mathbf{f}) \\
&=: (U_2^i - U_2^j)^2 \left(\mathcal{C}_{12}^{ij} \delta^{-3/2} + \mathcal{C}_{13}^{ij} \delta^{-1/2} \right) \\
&\quad + \mathcal{C}_{14}^{ij} (U_2^i - U_2^j) Ra \delta^{-1/2} + E_2^b(\mathbf{U}^i, \mathbf{f}).
\end{aligned} \tag{B.33}$$

where the error term

$$E_2^b(\mathbf{U}^i, \mathbf{f}) = \mu C_{11} [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{q}^{ij}]^2, \tag{B.34}$$

with universal constant C_{11} and

$$\mathcal{C}_{12}^{ij} = 6\pi\mu R^{3/2} d_{ij}^{-3/2}, \mathcal{C}_{13}^{ij} = \frac{63}{20} \pi\mu R^{1/2} d_{ij}^{-1/2}, \mathcal{C}_{14}^{ij} = 6\pi\mu R^{1/2} d_{ij}^{-1/2}.$$

When we evaluate the difference between $W(\mathbf{v}_2)$, defined by (B.33), and $W_2^*(\mathcal{S}_2)$, defined by (6.13) we obtain the following estimate:

$$|W(\mathbf{v}_2) - W_2^*(\mathcal{S}_2)| \leq E_2^b(\mathbf{U}^i, \mathbf{f}), \quad \text{as } \delta \rightarrow 0. \tag{B.35}$$

Therefore, $W(\mathbf{v}_2)$ determines the asymptotics of $W(\mathbf{u}_2)$ in the neck connecting the disk and quasidisk:

$$\begin{aligned}
W(\mathbf{u}_2) &= [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{q}^{ij}]^2 \left(\mathcal{B}_{12}^{ij} \delta^{-3/2} + \mathcal{B}_{13}^{ij} \delta^{-1/2} \right) \\
&\quad + \mathcal{B}_{14}^{ij} [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{q}^{ij}] Ra \delta^{-1/2} + E_2^b(\mathbf{U}^i, \mathbf{f}).
\end{aligned} \tag{B.36}$$

C Appendix

1 Positive definiteness of Q_{ij} in (2.36)

Lemma C.1. *The quadratic form Q_{ij} , defined by (2.36), (A.1), is positive definite.*

We show the positive definiteness of Q_{ij} with coefficients (A.1) for $i \in \mathbb{I}$, $j \in \mathcal{N}_i \cap \mathbb{I}$ (the case when $j \in \mathcal{N}_i \cap \mathbb{B}$ can be treated similarly). To this end we just need to show that $Q_{ij}(\mathbf{U}^i, \mathbf{U}^j, \omega^i, \omega^j, \beta_{ij})$ can be written as a sum of squares. Indeed,

$$\begin{aligned}
Q(\mathbf{U}^i, \mathbf{U}^j, \omega^i, \omega^j, \beta_{ij}) &= \frac{\pi\mu\sqrt{R}}{\sqrt{d_{ij}}\sqrt{\delta}} \left\{ \frac{1}{2} [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij} + R\omega^i + R\omega^j]^2 \right. \\
&\quad + [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{q}^{ij}]^2 \left(\frac{3}{8}R\delta^{-1} + \frac{207}{320} \right) \\
&\quad + \beta_{ij}^2 \left(2\delta^{-2} + \frac{249}{160R}\delta^{-1} + \frac{13491}{17920R^2} \right) \\
&\quad + \left[\frac{3}{4}R(\omega^i - \omega^j) - \beta_{ij} \left(\frac{1}{2}\delta^{-1} - \frac{15}{16R} \right) \right]^2 \Big\} \\
&\quad + E_{ij}(\mathbf{U}^i, \mathbf{U}^j, \omega^i, \omega^j, \beta_{ij}) \\
&= \frac{\pi\mu\sqrt{R}}{\sqrt{d_{ij}}\sqrt{\delta}} \left\{ \frac{1}{2} [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{p}^{ij} + R\omega^i + R\omega^j]^2 \right. \\
&\quad + [(\mathbf{U}^i - \mathbf{U}^j) \cdot \mathbf{q}^{ij}]^2 \left(\frac{3}{8}R\delta^{-1} + \frac{207}{320} \right) \\
&\quad + \beta_{ij}^2 \left(\sqrt{2}\delta^{-1} + \frac{249}{320\sqrt{2}R} \right)^2 + \frac{645273}{1433600R^2}\beta_{ij}^2 \\
&\quad + \left[\frac{3}{4}R(\omega^i - \omega^j) - \beta_{ij} \left(\frac{1}{2}\delta^{-1} - \frac{15}{16R} \right) \right]^2 \Big\} \\
&\quad + E_{ij}(\mathbf{U}^i, \mathbf{U}^j, \omega^i, \omega^j, \beta_{ij})
\end{aligned}$$

2 Existence and uniqueness of the solution to (2.4)

Lemma C.2. *There exists a unique minimizer of $W_{\Omega_F}(\cdot)$ over V .*

Proof. First of all we note that V is not empty. Indeed, let \mathbf{w} be a solution to Stokes equation satisfying the Dirichlet boundary conditions, that is, of

$$\begin{cases} (a) \mu \Delta \mathbf{u} = \nabla p, & \mathbf{x} \in \Omega_F \\ (b) \nabla \cdot \mathbf{u} = 0, & \mathbf{x} \in \Omega_F \\ (c) \mathbf{u} = \mathbf{U}_0^i + R\omega_0^i(n_1^i \mathbf{e}_2 - n_2^i \mathbf{e}_1), \mathbf{x} \in \partial B^i, & i = 1, \dots, N \\ (d) \mathbf{u} = \mathbf{f}, & \mathbf{x} \in \partial\Omega \end{cases} \quad (\text{C.1})$$

with some given \mathbf{U}_0^i and ω_0^i , $i = 1, \dots, N$, which is known to exist. Then $\mathbf{w} \in V$.

The existence of the minimizer \mathbf{u} would be shown in few steps.

Step 1. Set $m := \inf_V W_{\Omega_F}(\cdot)$. If $m = +\infty$ then we are done. So we henceforth assume that m is finite. Select a minimizing sequence $\{\mathbf{u}_k\}_{k=1}^\infty$ of the functional $W_{\Omega_F}(\cdot)$, that is, $W_{\Omega_F}(\mathbf{u}_k) \rightarrow m$ as $k \rightarrow \infty$.

Step 2. Choose a sequence $\{\bar{\mathbf{u}}_k\}_{k=1}^\infty$, where each $\bar{\mathbf{u}}_k$ solves a Stokes equation (C.1) with boundary conditions (C.1c) on ∂B^i which are the same as the corresponding boundary conditions of \mathbf{u}_k . Hence,

$$m \leq W_{\Omega_F}(\bar{\mathbf{u}}_k) \leq W_{\Omega_F}(\mathbf{u}_k) \rightarrow m \quad \text{as } k \rightarrow \infty.$$

Now we claim that there exists a convergent subsequence $\{\varphi_{kj}^i\}$ of values of velocities on each disk, that is, there exists φ^i such that $\varphi_{kj}^i \rightarrow \varphi^i$ as $j \rightarrow \infty$ on each ∂B^i , $i = 1, \dots, N$

We prove this assertion by contradiction. Assume there exists such a disk B^{i_0} and a subsequence $\{\varphi_{kj}^{i_0}\} = \{\mathbf{U}_{kj}^{i_0} + \boldsymbol{\Omega}_{kj}^{i_0}\}$ on it that $\varphi_{kj}^{i_0} \rightarrow \infty$ as $j \rightarrow \infty$, where $\mathbf{U}_{kj}^{i_0}$ is the translational velocity and $\boldsymbol{\Omega}_{kj}^{i_0}$ is the rotational part. We select a subsequence $\{\bar{\mathbf{u}}_{kj}\} \in V$ of the above sequence $\{\bar{\mathbf{u}}_k\}$, that is, $\{\bar{\mathbf{u}}_{kj}\}$ solves the Stokes problem with fixed values φ_{kj}^i on the disks B^i . Since $W_{\Omega_F}(\bar{\mathbf{u}}_{kj})$ is bounded and $\varphi_{kj}^{i_0} \rightarrow \infty$ we have

$$W_{\Omega_F} \left(\frac{\bar{\mathbf{u}}_{kj}}{\varphi_{kj}^{i_0}} \right) = \frac{1}{(|\varphi_{kj}^{i_0}|)^2} W_{\Omega_F}(\bar{\mathbf{u}}_{kj}) \leq \varepsilon_{kj}^{i_0},$$

where $\varepsilon_{kj}^{i_0}$ is a small arbitrary number.

Without loss of generality, we assume that the disk B^{i_0} lies in the corner of the domain Ω , that is, near two parts of the boundary $\partial\Omega$ (see Fig. C.1), where values of $\bar{\mathbf{u}}_{kj}$ are prescribed and equal to \mathbf{f} .

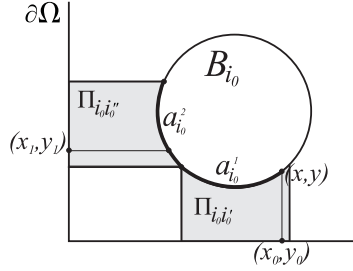


Fig. C.1.

First consider a neck $\Pi_{i_0 i_0'}$ and choose a symmetric neck $\tilde{\Pi}_{i_0 i_0'}$ containing $\Pi_{i_0 i_0'}$, that is, $\tilde{\Pi}_{i_0 i_0'} \supset \Pi_{i_0 i_0'}$. For simplicity we drop indices kj indicating the subsequence and i_0 indicating the disk number. Then denote $\boldsymbol{\varphi} = (\varphi_1, \varphi_2) = (\mathbf{U}, \boldsymbol{\Omega}) = (U_1 + \Omega_1, U_2 + \Omega_2)$ and $\frac{\bar{\mathbf{u}}}{|\boldsymbol{\varphi}|} = (\bar{u}, \bar{v})$. Choose a point (x_0, y_0) on the lower part of $\partial\Omega$ and (x, y) on ∂B^{i_0} (Fig. C.1). By integrating along the line connecting x and x_0 we obtain:

$$\bar{v}(x, y) = \frac{f_2(x_0, y_0)}{|\boldsymbol{\varphi}|} + \int_{y_0}^y \frac{\partial \bar{v}}{\partial y}(x, y) dy,$$

or

$$|\bar{v}(x, y)| \leq \eta_2 + \int_0^{H(\gamma_{i_0 i_0'})/2} \left| \frac{\partial \bar{v}}{\partial y}(x, y) \right| dy,$$

where $\bar{v}(x, y) = \varphi_2$ and η_2 is some small number since $\frac{f_2(x_0, y_0)}{|\varphi|} \rightarrow 0$ as $j \rightarrow \infty$ and $H(\gamma_{i_0 i'_0})/2$ is a half-height of the neck $\tilde{\Pi}_{i_0 i'_0}$. Then integrate over the arc $a_{i_0}^1$ of ∂B^{i_0} which is an upper boundary boundary of the neck $\Pi_{i_0 i'_0}$:

$$\left| \int_{a_{i_0}^1} \bar{v} ds \right| \leq \int_{a_{i_0}^1} |\bar{v}| ds \leq |a_{i_0}^1| \eta_2 + \int_{\Pi_{i_0 i'_0}} \left| \frac{\partial \bar{v}}{\partial y}(x, y) \right| d\mathbf{x}$$

and

$$\left| \int_{a_{i_0}^1} \bar{v} ds \right|^2 \leq C_1 \eta_2 + C_2 \int_{\Pi_{i_0 i'_0}} \left| \frac{\partial \bar{v}}{\partial y}(x, y) \right|^2 d\mathbf{x} \leq C_1 \eta_2 + C_2 W_{\Omega_F}(\bar{\mathbf{u}}) \leq C_1 \eta_2 + C_2 \varepsilon.$$

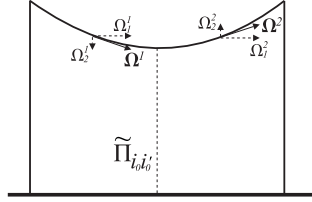


Fig. C.2.

Consider

$$\int_{a_{i_0}^1} \bar{v} ds = \int_{a_{i_0}^1} (U_2 + \Omega_2) ds = |a_{i_0}^1| U_2 + \int_{a_{i_0}^1} \Omega_2 ds.$$

Note that $\int_{a_{i_0}^1} \Omega_2 ds = 0$ since the second components of the vector Ω cancel each other when we write an integral as a sum over infinitesimal elements of the symmetric arc $a_{i_0}^1$ (see Fig. C.2). Hence,

$$C_3 U_2^2 \leq C_1 \eta_2 + C_2 \varepsilon,$$

where $\eta_2 = (\eta_2)_{kj}^{i_0}$ and $\varepsilon = \varepsilon_{kj}^{i_0}$ are small arbitrary numbers. So the second component of the vector $\mathbf{U}_{kj}^{i_0}$ tends to zero as $j \rightarrow \infty$.

Similarly, one can show that the first component of the vector $\mathbf{U}_{kj}^{i_0}$ tends to zero as $j \rightarrow \infty$ considering the neck $\Pi_{i_0 i''_0}$.

Thus, we prove that $\mathbf{U}_{kj}^{i_0} \rightarrow 0$ as $j \rightarrow \infty$.

To show that $\Omega_{kj}^{i_0} \rightarrow 0$ as $j \rightarrow \infty$ we consider only the left-hand side of the neck $\Pi_{i_0 i'_0}$. Repeating all computations described above, we obtain that the integral of Ω_2 over the left half of the symmetric arc $a_{i_0}^{1\ell}$ tends to zero as $j \rightarrow \infty$:

$$\int_{a_{i_0}^{1\ell}} \Omega_2 ds \leq C_1 \eta_2 + C_2 \varepsilon,$$

But $\Omega_2 = \omega_{kj}^{i_0} \cos \theta$, where $\omega_{kj}^{i_0}$ is a constant and θ is an angle of the vector $\Omega_{kj}^{i_0}$ with vertical axis. Thus, $\theta > 0$ everywhere in that half of the neck, hence $\Omega_2 \rightarrow 0$. Similarly, one can show that $\Omega_1 \rightarrow 0$, hence, the whole subsequence $\varphi_{kj}^{i_0}$ is convergent what contradicts our assumption.



Fig. C.3.

Note that if B^{i_0} is not near $\partial\Omega$ one can use the same argument using two nearest neighboring disks B^{m_1} and B^{m_2} of B^{i_0} where there exist convergent subsequences $\varphi_{kj}^{m_1}$ and $\varphi_{kj}^{m_2}$, respectively. We can always assume that since our graph is connected, that there exist paths connecting the given disk with all parts of the boundary $\partial\Omega$.

Step 3. Denote the limit of convergent subsequences on each disk B^i by \mathbf{v}^i , $i = 1, \dots, N$ and consider the subsequence $\{\bar{\mathbf{u}}_{kj}\} \in V$ such that on the boundary of each disk ∂B^i : $\bar{\mathbf{u}}_{kj} \rightarrow \mathbf{v}^i$ as $j \rightarrow \infty$.

Step 4. Choose a sequence $\{\bar{\bar{\mathbf{u}}}_j\} \in V$ such that each $\bar{\bar{\mathbf{u}}}_j$ solves the Stokes problem (4.5) with $\bar{\bar{\mathbf{u}}}_j = \mathbf{v}^i$ on ∂B^i , $i = 1, \dots, N$.

Step 5. Note that

$$W_{\Omega_F}(\bar{\bar{\mathbf{u}}}_j) \leq W_{\Omega_F}(\bar{\mathbf{u}}_{kj}) + W_{\Omega_F}(\hat{\mathbf{u}}_j) + 2\sqrt{W_{\Omega_F}(\bar{\mathbf{u}}_{kj})W_{\Omega_F}(\hat{\mathbf{u}}_j)}, \quad (\text{C.2})$$

where $\hat{\mathbf{u}}_j = \bar{\mathbf{u}}_{kj} - \bar{\bar{\mathbf{u}}}_j$, that is solves

$$\begin{cases} (a) \mu \Delta \mathbf{u} = \nabla p, & \mathbf{x} \in \Omega_F \\ (b) \nabla \cdot \mathbf{u} = 0, & \mathbf{x} \in \Omega_F \\ (c) \mathbf{u} = \mathbf{v}_{kj}^i - \mathbf{v}^i, & \mathbf{x} \in \partial B^i, \quad i = 1, \dots, N \\ (d) \mathbf{u} = \mathbf{0}, & \mathbf{x} \in \partial\Omega \end{cases}$$

Hence, (see e.g. [26]) $\|\hat{\mathbf{u}}_j\|_{H^1(\Omega_F)} \leq C \|\mathbf{v}_{kj}^i - \mathbf{v}^i\|_{H^{1/2}(\partial B^i)} \rightarrow 0$ as $j \rightarrow \infty$. Then from (C.2) we obtain

$$W_{\Omega_F}(\bar{\bar{\mathbf{u}}}_j) \rightarrow m \quad \text{as } j \rightarrow \infty.$$

Step 6. Choose $\bar{\mathbf{u}} = \bar{\mathbf{u}}_1$ and $\mathbf{u}_j^0 = \bar{\mathbf{u}}_j - \bar{\mathbf{u}}$ where $\mathbf{u}_j^0 \in H_0^1(\Omega_F)$ such that $W_{\Omega_F}(\mathbf{u}_j^0 + \bar{\mathbf{u}}) \rightarrow m$ as $j \rightarrow \infty$.

Step 7. Consider

$$\begin{aligned} \|\bar{\mathbf{u}}_j\|_{L^2(\Omega_F)} &\leq \|\bar{\mathbf{u}}_j - \bar{\mathbf{u}}\|_{L^2(\Omega_F)} + \|\bar{\mathbf{u}}\|_{L^2(\Omega_F)} \\ &= \|\mathbf{u}_j^0\|_{L^2(\Omega_F)} + \|\bar{\mathbf{u}}\|_{L^2(\Omega_F)} \leq C_1 \|\nabla \mathbf{u}_j^0\|_{L^2(\Omega_F)} + C_2 \end{aligned} \quad (\text{C.3})$$

Note that for the $H_0^1(\Omega_F)$ -function \mathbf{u}_j^0 one has

$$\begin{aligned} W_{\Omega_F}(\mathbf{u}_j^0) &= \mu \int_{\Omega_F} \left\{ \left(\frac{\partial u_{j1}^0}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial u_{j1}^0}{\partial y} + \frac{\partial u_{j2}^0}{\partial x} \right)^2 + \left(\frac{\partial u_{j2}^0}{\partial y} \right)^2 \right\} d\mathbf{x} \\ &= \mu \int_{\Omega_F} \left\{ \left(\frac{\partial u_{j1}^0}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial u_{j1}^0}{\partial y} \right)^2 + \frac{1}{2} \left(\frac{\partial u_{j2}^0}{\partial x} \right)^2 + \frac{\partial u_{j1}^0}{\partial y} \frac{\partial u_{j2}^0}{\partial x} + \left(\frac{\partial u_{j2}^0}{\partial y} \right)^2 \right\} d\mathbf{x} \\ &= \mu \int_{\Omega_F} \left\{ \left(\frac{\partial u_{j1}^0}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial u_{j1}^0}{\partial y} \right)^2 + \frac{1}{2} \left(\frac{\partial u_{j2}^0}{\partial x} \right)^2 + \frac{\partial u_{j1}^0}{\partial x} \frac{\partial u_{j2}^0}{\partial y} + \left(\frac{\partial u_{j2}^0}{\partial y} \right)^2 \right\} d\mathbf{x} \\ &= \mu \int_{\Omega_F} \left\{ \frac{1}{2} \left(\frac{\partial u_{j1}^0}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial u_{j1}^0}{\partial y} \right)^2 + \frac{1}{2} \left(\frac{1}{2} \frac{\partial u_{j2}^0}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial u_{j2}^0}{\partial y} \right)^2 \right\} d\mathbf{x} \\ &= \frac{\mu}{2} \int_{\Omega_F} |\nabla \mathbf{u}_j^0|^2 d\mathbf{x} = \frac{\mu}{2} \|\nabla \mathbf{u}_j^0\|_{L^2(\Omega_F)}^2 \end{aligned}$$

Here we used the integration by parts (all boundary terms disappear since $\mathbf{u}_j^0 \in H_0^1(\Omega_F)$) and the divergence free property of \mathbf{u}_j^0 . Thus, continuing (C.3) we obtain

$$\begin{aligned} C_1 \|\nabla \mathbf{u}_j^0\|_{L^2(\Omega_F)} + C_2 &\leq C_3 W_{\Omega_F}(\mathbf{u}_j^0) + C_2 \\ &\leq C_3 \left(W_{\Omega_F}(\bar{\mathbf{u}}_j) + W_{\Omega_F}(\bar{\mathbf{u}}) + 2\sqrt{W_{\Omega_F}(\bar{\mathbf{u}}_j)W_{\Omega_F}(\bar{\mathbf{u}})} \right) + C_2 \leq C_4 \end{aligned}$$

So the sequence $\bar{\mathbf{u}}_j$ is bounded in $L^2(\Omega_F)$, as well as

$$\|\nabla \bar{\mathbf{u}}_j\|_{L^2(\Omega_F)} \leq \|\nabla \mathbf{u}_j^0\|_{L^2(\Omega_F)} + \|\nabla \bar{\mathbf{u}}\|_{L^2(\Omega_F)} \leq C_5.$$

Note that $\bar{\mathbf{u}} \in H^1(\Omega_F)$ as a solution to the Stokes problem [26].

Therefore, from all the above $\bar{\mathbf{u}}_j$ is bounded sequence in $H^1(\Omega_F)$.

Step 8. From the previous step it follows that there exists a vector field $\mathbf{u} \in H^1(\Omega_F)$ such that $\bar{\mathbf{u}}_j \rightharpoonup \mathbf{u}$ (weakly) in $H^1(\Omega_F)$ as $j \rightarrow \infty$.

We have to show that $\mathbf{u} \in V$. Recall that for $\bar{\mathbf{u}} \in V$: $\bar{\mathbf{u}}_j - \bar{\mathbf{u}} \in H_0^1(\Omega_F)$, which is closed linear subspace of $H^1(\Omega_F)$, hence, $H_0^1(\Omega_F)$ is weakly closed (Mazur's theorem), thus

$$\bar{\mathbf{u}}_j - \bar{\mathbf{u}} \rightharpoonup \mathbf{u} - \bar{\mathbf{u}},$$

therefore, $\mathbf{u} \in V$.

Step 9. Finally, we show that $W_{\Omega_F}(\mathbf{u}) = m$. The functional $W_{\Omega_F}(\cdot)$ is convex in $\nabla \mathbf{u}$ hence it is weakly lower semi-continues, that is,

$$W_{\Omega_F}(\mathbf{u}) \leq \liminf_{j \rightarrow \infty} \bar{\mathbf{u}}_j = m.$$

Since $\mathbf{u} \in V$, we have

$$W_{\Omega_F}(\mathbf{u}) = m = \min_V W_{\Omega_F}(\cdot).$$

3 Existence and uniqueness of the solution to (2.10)

Lemma C.3. *There exists a unique minimizer of $W_{\Pi}(\cdot)$ over V_{Π} .*

Proof. To show the existence of a minimizer of functional $W_{\Pi}(\mathbf{v})$ for $\mathbf{v} \in V_{\Pi}$ we have to prove that it is coercive and convex in the gradient of \mathbf{v} and that V_{Π} is nonempty convex and closed set.

Step 1. Set $m := \inf_{V_{\Pi}} W_{\Pi}$. If $m = +\infty$, we are done. So we hereafter assume that m is finite. Select a minimizing sequence $\{\mathbf{u}_n\}_{n=1}^{\infty} \subset V_{\Pi}$ of $W_{\Pi}(\cdot)$, that is, $W_{\Pi}(\mathbf{u}_n) \rightarrow m$ as $n \rightarrow \infty$.

Step 2. By the argument similar to Step 2 of Lemma C.2 one can choose a bounded subsequence of the velocities (which are constant vectors) on the disks B^i , $i = 1, \dots, N$. Denote such a subsequence by $\{\mathbf{v}_n^i\}_{n=1}^{\infty}$, $i = 1, \dots, N$, that is, there exists a positive C_0 such that for all n and every $i = 1, \dots, N$: $|\mathbf{v}_n^i| < C_0$. Denote the limits of those subsequences by \mathbf{v}^i .

Note that by the Korn's inequality (see e.g. [18]) we have:

$$\|\nabla \mathbf{u}_n\|_{L^2(\Pi)}^2 \leq C_{\Pi} \left(W_{\Pi}(\mathbf{u}_n) + \|\mathbf{u}_n\|_{L^2(\Pi)}^2 \right), \quad (\text{C.4})$$

where C_{Π} is some constant. Therefore, we first show that the sequence \mathbf{u}_n is bounded in $L^2(\Pi)$.

Step 3. Select a sequence \mathbf{w}_n of solutions to the Stokes problem (C.1) with $\mathbf{w}_n = \mathbf{v}_n^i$ on each ∂B^i , and the functions \mathbf{w} that solves (C.1) with $\mathbf{w} = \mathbf{v}^i$ on ∂B^i , $i = 1, \dots, N$. Then,

$$\|\mathbf{w}_n\|_{H^1(\Omega_F)} \leq \|\mathbf{w}_n - \mathbf{w}\|_{H^1(\Omega_F)} + \|\mathbf{w}\|_{H^1(\Omega_F)},$$

where both norms of the right-hand side are bounded [26] as solutions to Dirichlet problem, thus, $\|\mathbf{w}_n\|_{H^1(\Pi)} \leq \mathcal{C}$.

Define a sequence $\varphi_n = \mathbf{w}_n|_{\Pi} - \mathbf{u}_n$ in Π , which is zero vectors on each sphere ∂B^i and zero vector on the external boundary $\partial \Omega$. Using (C.4) for the sequence φ_n and the fact that

$$\begin{aligned} W_{\Pi}(\varphi_n) &\leq W_{\Pi}(\mathbf{w}_n) + W_{\Pi}(\mathbf{u}_n) + 2\sqrt{W_{\Pi}(\mathbf{w}_n)W_{\Pi}(\mathbf{u}_n)} \\ &\leq \mathcal{C} + (m + \varepsilon_n) + 2\sqrt{\mathcal{C}(m + \varepsilon_n)} \leq C_1, \end{aligned}$$

we prove that φ_n is bounded in $L^2(\Pi)$.

Consider the neck Π_{ij} (Fig. C.4) between adjacent disks B^i and B^j . Choose a point (x_0, y_0) on the sphere ∂B^j and $(x, y) \in \Pi_{ij}$, and consider

$$\varphi_n^1(x, y) = \int_{x_0}^x \frac{\partial \varphi_n^1}{\partial \xi}(\xi, y) d\xi,$$

for almost all x , where the superscript here indicates a component of the the vector $\varphi_n = (\varphi_n^1, \varphi_n^2)$. Then we obtain the following estimate:

$$|\varphi_n^1(x, y)|^2 \leq C_2 \int_{x_0}^{H(\gamma)/2} \left| \frac{\partial \varphi_n^1}{\partial \xi}(\xi, y) \right|^2 d\xi,$$

and $\gamma = \max(\gamma_{ij}^-, \gamma_{ij}^+)$. Now we integrate the last inequality with respect to y between γ_{ij}^- and γ_{ij}^+ , and with respect to x along the arc a_i obtaining

$$\int_{\Pi_{ij}} |\varphi_n^1(x, y)|^2 d\mathbf{x} = \|\varphi_n^1(x, y)\|_{L^2(\Pi_{ij})}^2 \leq C_2 |\gamma_{ij}^+ - \gamma_{ij}^-| H(\gamma)/2 = C_3.$$

Here we recall that $\int_{\Pi_{ij}} \left| \frac{\partial \varphi_n^1}{\partial \xi} \right|^2 d\mathbf{x}$ is bounded from above by $W_{\Pi}(\varphi_n)$. So the first components of φ_n form a bounded sequence in $L^2(\Pi_{ij})$ for any neck $\Pi_{ij} \in \Pi$.

To show that the second components of φ_n is a bounded sequence we consider another neck Π_{ki} (see Fig. C.4(b)) that forms a small angle α with the direction of the second component φ_n^2 . Then by triangle inequality we have

$$\|\varphi_n^2\|_{L^2(\Pi_{ik})} \leq C(\alpha) (\|\varphi_n^1\|_{L^2(\Pi_{ik})} + \|\varphi_n^3\|_{L^2(\Pi_{ik})}),$$

where the constant $C(\alpha)$ depends on the small angle α . Since the viscous dissipation rate is invariant under rotation we evaluate the norm of φ_n^3 to obtain an estimate for φ_n^2 .

In the system of coordinate with y -axis along the neck Π_{ki} we consider

$$\varphi_n^3(x, y) = \int_{y_0}^y \frac{\partial \varphi_n^3}{\partial \eta}(x, \eta) d\eta,$$

where the point (x_0, y_0) lies on ∂B^k . As before, integrating with respect to x over $(\gamma_{ki}^-, \gamma_{ki}^+)$ and with respect to y over the arc a_k and using that $\int_{\Pi_{ij}} \left| \frac{\partial \varphi_n^3}{\partial \eta} \right|^2 d\mathbf{x}$ is bounded from above by $W_{\Pi}(\varphi_n)$, we obtain the following estimate:

$$\int_{\Pi_{ij}} |\varphi_n^3(x, y)|^2 d\mathbf{x} = \|\varphi_n^3(x, y)\|_{L^2(\Pi_{ij})}^2 \leq C_4.$$

Hence, the sequence φ_n is bounded in $L^2(\Pi)$. In the view of the Korn's inequality (C.4) we obtain that φ_n is bounded in $H^1(\Pi)$. By triangle inequality we obtain that sequence \mathbf{u}_n is bounded in $H^1(\Pi)$ as well.

The boundedness of \mathbf{u}_n in $H^1(\Pi)$ implies there exists a subsequence $\mathbf{u}_{nn'}$ and a function $\hat{\mathbf{u}} \in H^1(\Pi_{ij})$ such that $\mathbf{u}_{nn'} \rightharpoonup \hat{\mathbf{u}}$ (weakly) in $H^1(\Pi_{ij})$ as

$n' \rightarrow \infty$. It remains to show that $\hat{\mathbf{u}} \in V_{\mathbf{\Pi}}$ and $W_{\mathbf{\Pi}}(\hat{\mathbf{u}}) = m$. To do so we need to prove that $V_{\mathbf{\Pi}}$ is convex and closed set of $H^1(\mathbf{\Pi})$, and $W_{\mathbf{\Pi}}(\cdot)$ is a convex functional.

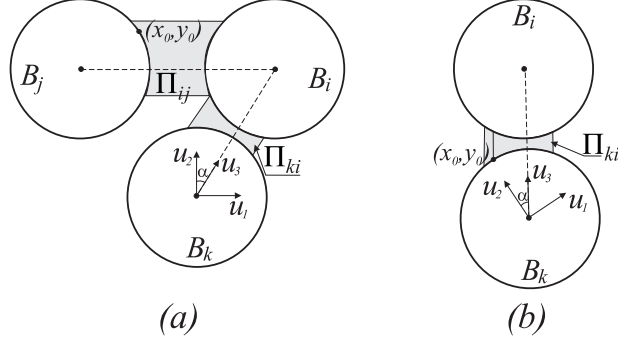


Fig. C.4.

Step 4. $V_{\mathbf{\Pi}}$ is convex if for any $\mathbf{u}, \mathbf{v} \in V_{\mathbf{\Pi}}$ their convex combination is from V_{ij} : $\mathbf{w} = (1 - \lambda)\mathbf{u} + \lambda\mathbf{v} \in V_{\mathbf{\Pi}}$, $0 < \lambda < 1$. To show that $\mathbf{w} \in V_{\mathbf{\Pi}}$ we have to prove $\nabla \cdot \mathbf{w} = 0$ in $\mathbf{\Pi}$, $\mathbf{w} = \mathbf{f}$ on $\partial\mathbf{\Pi} \cap \partial\Omega$, $\mathbf{w} = \mathbf{U}^i + R\omega^i(n_1^i \mathbf{e}_2 - n_2^i \mathbf{e}_1)$ on ∂B^i and $\int_{\partial\Delta_{ijk}} \mathbf{w} \cdot \mathbf{n} ds = 0$, which all clearly hold.

Step 5. To prove that $V_{\mathbf{\Pi}}$ is closed we consider a sequence $\{\mathbf{v}_n\}_{n=1}^{\infty}$, $\mathbf{v}_n \in V_{\mathbf{\Pi}}$ for all n and $\mathbf{v}_n \rightarrow \mathbf{v}$ in $H^1(\mathbf{\Pi})$, and show that $\mathbf{v} \in V_{\mathbf{\Pi}}$ as well. For this we need to check that $\nabla \cdot \mathbf{v} = 0$ in $\mathbf{\Pi}$ in the weak sense, and $\mathbf{v} = \mathbf{U}^i + R\omega^i(n_1^i \mathbf{e}_2 - n_2^i \mathbf{e}_1)$ on ∂B^i , $i = 1, \dots, N$, $\int_{\partial\Delta_{ijk}} \mathbf{v} \cdot \mathbf{n} ds = 0$, and $\mathbf{v} = \mathbf{f}$ if $\partial\mathbf{\Pi} \cap \partial\Omega$, which understood in the sense of trace. Any convergent sequence in $H^1(\mathbf{\Pi})$ is weakly convergent, hence, $\int_{\mathbf{\Pi}} \mathbf{v}_n \cdot \nabla \varphi d\mathbf{x} \rightarrow \int_{\mathbf{\Pi}} \mathbf{v} \cdot \nabla \varphi d\mathbf{x}$ for every $\varphi \in C_0^\infty(\mathbf{\Pi})$. Thus the incompressibility of \mathbf{v} holds.

The boundary conditions of the definition of $V_{\mathbf{\Pi}}$ are considered in the sense of trace, that is, for the H^1 function \mathbf{v}_n there exist a linear bounded map $\gamma(\mathbf{v}_n)$ and a constant C such that $\|\gamma(\mathbf{v}_n)\|_{L_2(\partial\mathbf{\Pi})} \leq C\|\mathbf{v}_n\|_{H^1(\mathbf{\Pi})}$. Therefore, we have

$$\begin{aligned}
 & \sum_{i \in \mathbb{I}, j, k \in \mathcal{N}_i} \int_{\partial\Delta_{ijk}} (\gamma(\mathbf{v}_k) - \gamma(\mathbf{v})) \cdot \mathbf{n} ds = \sum_{i \in \mathbb{I}, j, k \in \mathcal{N}_i} \int_{\partial\Delta_{ijk}} \gamma(\mathbf{v}_n - \mathbf{v}) \cdot \mathbf{n} ds \\
 & \leq \sum_{i \in \mathbb{I}, j, k \in \mathcal{N}_i} |\partial\Delta_{ijk}|^{1/2} \|\gamma(\mathbf{v}_n - \mathbf{v})\|_{L_2(\partial\Delta_{ijk})}^{1/2} \leq C_1 \|\gamma(\mathbf{v}_n - \mathbf{v})\|_{L_2(\partial\mathbf{\Pi})}^{1/2} \\
 & \leq C_2 \|\mathbf{v}_n - \mathbf{v}\|_{H^1(\mathbf{\Pi})}^{1/2} \rightarrow 0 \text{ as } n \rightarrow \infty,
 \end{aligned}$$

where $C_2 = \left(C \sum_{i \in \mathbb{I}, j, k \in \mathcal{N}_i} |\partial \Delta_{ijk}| \right)^{1/2}$, and $\partial \mathbf{\Pi} = \bigcup_{i \in \mathbb{I}} \partial B^i \cup \bigcup_{i \in \mathbb{I}, j, k \in \mathcal{N}_i} \partial \Delta_{ijk}$

The remaining boundary conditions can be proved similarly.

Finally, we show that $W_{\mathbf{\Pi}}(\cdot)$ is convex in $\nabla \mathbf{u}$. For this we write

$$W_{\mathbf{\Pi}}(\nabla \mathbf{u}) = \mu \int_{\mathbf{\Pi}} L(\nabla \mathbf{u}) d\mathbf{x},$$

where $L(\mathbf{p}) = p_{11}^2 + \frac{1}{2}(p_{12} + p_{21})^2 + p_{22}^2$ and $\mathbf{p} = (p_{ij})_{i,j=1,2}$ is 2×2 matrix. Recall that by definition convexity means that

$$\sum_{i,j,k,l=1}^2 L_{p_{ij}p_{kl}}(\mathbf{p}) a_{ij} a_{kl} \geq 0, \quad \text{for any } A = (a_{ij})_{i,j=1,2}.$$

Therefore, we have

$$\begin{aligned} L_{p_{11}p_{11}} &= L_{p_{22}p_{22}} = 2, & L_{p_{12}p_{12}} &= L_{p_{21}p_{21}} = L_{p_{12}p_{21}} = L_{p_{21}p_{12}} = 1, \\ L_{p_{11}p_{12}} &= L_{p_{11}p_{21}} = L_{p_{11}p_{22}} = L_{p_{12}p_{11}} = L_{p_{12}p_{22}} = L_{p_{21}p_{11}} = L_{p_{21}p_{22}} = 0, \\ L_{p_{22}p_{11}} &= L_{p_{22}p_{12}} = L_{p_{22}p_{21}} = 0. \end{aligned}$$

Hence,

$$2a_{11}^2 + 2a_{22}^2 + a_{12}^2 + a_{21}^2 + a_{12}a_{12} + a_{21}a_{21} = 2a_{11}^2 + 2a_{22}^2 + (a_{12} + a_{21})^2 \geq 0,$$

thus, $W_{\mathbf{\Pi}}(\cdot)$ is convex in $\nabla \mathbf{u}$. This implies that $W_{\mathbf{\Pi}}$ is convex, it is weakly lower semicontinuous:

$$W_{\mathbf{\Pi}}(\hat{\mathbf{u}}) \leq \liminf_n W_{\mathbf{\Pi}}(\mathbf{u}_n) = m,$$

and $W_{\mathbf{\Pi}}(\hat{\mathbf{u}}) = m$.

By Mazur's Theorem $V_{\mathbf{\Pi}}$ is weakly closed, hence, $\hat{\mathbf{u}} \in V_{\mathbf{\Pi}}$.

Step 6. To show a uniqueness of the minimizer we proceed by contradiction. Assume that there exist two minimizers of this problem $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$. Consider their difference $\hat{\mathbf{w}} = \hat{\mathbf{u}} - \hat{\mathbf{v}}$ that solves (4.5) with $\mathbf{f} = \mathbf{0}$ on $\partial \Omega$ and some $\hat{\mathbf{W}}^i + R\omega_{\hat{\mathbf{w}}}^i(n_1^i e_2 - n_2^i e_1)$ on ∂B^i , $i = 1, \dots, N$. By multiplying equation (4.5a) by $\hat{\mathbf{w}}$ and integrating by parts over $\mathbf{\Pi}$ one obtains that $W_{\mathbf{\Pi}}(\hat{\mathbf{w}}) = 0$.

Using the argument given by Step 2 of Lemma C.2 we can show that $\hat{\mathbf{w}} = \mathbf{0}$ on each disk B^i , $i = 1, \dots, N$ and the argument of Step 3 of the current lemma we can prove that $\hat{\mathbf{w}} = \mathbf{0}$ in $\mathbf{\Pi}$.

4 Existence and uniqueness of the solution to (4.4)

Suppose there are two solutions to (4.4). Then their difference satisfies

$$(C.5) \quad \begin{cases} (a) \mu \Delta \mathbf{u} = \nabla p, & \mathbf{x} \in \Pi_{ij}, \\ (b) \nabla \cdot \mathbf{u} = 0, & \mathbf{x} \in \Pi_{ij}, \\ (c') \mathbf{u} = 0, & \mathbf{x} \in \partial B^i, \\ (c'') \mathbf{u} = 0, & \mathbf{x} \in \partial B^j, \\ (d) \frac{1}{R} \int_{\ell_{ij}} \mathbf{u} \cdot \mathbf{n} ds = 0, \\ (e) \boldsymbol{\sigma}(\mathbf{u}) \mathbf{n} = -p_{ij}^{\pm} \mathbf{n}, & \mathbf{x} \in \partial \Pi_{ij}^{\pm}, \\ (f) \mathbf{u} = 0, & \mathbf{x} \in \partial \Pi_{ij} \cap \partial \Omega. \end{cases}$$

Integrating by parts the incompressibility condition (C.5)(b) over half-necks and using (C.5)(b),(c'),(c''),(d),(e), we obtain:

$$\int_{\partial \Pi_{ij}^+} \mathbf{u} \cdot \mathbf{n} ds = \int_{\partial \Pi_{ij}^-} \mathbf{u} \cdot \mathbf{n} ds = 0.$$

Multiplying (C.5)(a) by \mathbf{u} and integrating by parts, we obtain

$$W_{\Pi_{ij}}(\mathbf{u}) = \int_{\partial \Pi_{ij}} \mathbf{u} \cdot \boldsymbol{\sigma}(\mathbf{u}) \mathbf{n} ds = p^+ \int_{\partial \Pi_{ij}^+} \mathbf{u} \cdot \mathbf{n} ds + p^- \int_{\partial \Pi_{ij}^-} \mathbf{u} \cdot \mathbf{n} ds = 0.$$

Hence \mathbf{u} must represent a rigid-body rotation. Due to our homogeneous boundary conditions, this motion must be identically zero.

Note, that since \mathbf{u} the solution to (C.5) must be identically zero, it implies that $p_{ij}^{\pm} = 0$. Hence we showed uniqueness of p_{ij}^{\pm} as well.

5 Dual problem to (2.10)

Lemma C.4. *Let $\mathbf{u}_{per} = (u_1^{per}, u_2^{per})$ is the minimizer of $W_{\Pi_{ij}}(\cdot)$ over V_{per} . Then the stress tensor $\boldsymbol{\sigma}(\mathbf{u}_{per})$ is the maximizer of $W_3^*(\cdot)$, defined by (6.14), over the set F_{per} , defined by (6.16).*

Proof. For simplicity we use a notation $\boldsymbol{\sigma}$ for $\boldsymbol{\sigma}(\mathbf{u}_{per})$. To prove that $\boldsymbol{\sigma}$ is a maximizer of $W_3^*(\cdot)$ we consider an arbitrary tensor $\mathbf{T} \in F_{per}$, such that $\mathbf{T} \mathbf{n} = \chi_T^{\pm} \mathbf{n}$ on $\partial \Pi_{ij}^{\pm}$, where χ_T^+ , χ_T^- are arbitrary numbers, then evaluate $W_3^*(\boldsymbol{\sigma} + \mathbf{T})$ and show that $W_3^*(\boldsymbol{\sigma}) \geq W_3^*(\boldsymbol{\sigma} + \mathbf{T})$.

Denote the first linear part of (6.14) by

$$\begin{aligned} \mathcal{L}(\boldsymbol{\sigma}) &= \frac{R\beta_{ij}^*}{H(\gamma_{ij}^+)} \int_{\partial \Pi_{ij}^+} \mathbf{n} \cdot \boldsymbol{\sigma} \mathbf{n} ds - \frac{R\beta_{ij}^*}{H(\gamma_{ij}^-)} \int_{\partial \Pi_{ij}^-} \mathbf{n} \cdot \boldsymbol{\sigma} \mathbf{n} ds \\ &= \frac{R\beta_{ij}^*}{H(\gamma_{ij}^+)} \int_{\partial \Pi_{ij}^+} \mathbf{n} \cdot (p_{ij}^+) \mathbf{n} ds - \frac{R\beta_{ij}^*}{H(\gamma_{ij}^-)} \int_{\partial \Pi_{ij}^-} \mathbf{n} \cdot (p_{ij}^-) \mathbf{n} ds \\ &= R\beta_{ij}^* \left(\frac{1}{H(\gamma_{ij}^+)} p_{ij}^+ |\partial \Pi_{ij}^+| - \frac{1}{H(\gamma_{ij}^-)} p_{ij}^- |\partial \Pi_{ij}^-| \right) \\ &= R\beta_{ij}^* (p_{ij}^+ - p_{ij}^-), \end{aligned}$$

while

$$\begin{aligned}\mathcal{L}(\mathbf{T}) &= \frac{R\beta_{ij}^*}{H(\gamma_{ij}^+)} \int_{\partial\Pi_{ij}^+} \mathbf{n} \cdot \mathbf{T} \mathbf{n} ds - \frac{R\beta_{ij}^*}{H(\gamma_{ij}^-)} \int_{\partial\Pi_{ij}^-} \mathbf{n} \cdot \mathbf{T} \mathbf{n} ds \\ &= R\beta_{ij}^*(\chi_T^+ - \chi_T^-).\end{aligned}$$

Also denote

$$\mathcal{M}(\boldsymbol{\sigma}) = R(\omega^i - \omega^j) \left[\int_{\partial B^i} \mathbf{G}_3^i \cdot \boldsymbol{\sigma} \mathbf{n} ds + \int_{\partial B^j} \mathbf{G}_3^j \cdot \boldsymbol{\sigma} \mathbf{n} ds \right].$$

Now we consider the quadratic term of (6.14) and notice that for $\boldsymbol{\sigma} + \mathbf{T}$ one has

$$\begin{aligned}& -\frac{1}{4\mu} \int_{\Pi_{ij}} \left[(\boldsymbol{\sigma} + \mathbf{T}) : (\boldsymbol{\sigma} + \mathbf{T}) - \frac{\text{tr}(\boldsymbol{\sigma} + \mathbf{T})^2}{2} \right] d\mathbf{x} \\ &= -\frac{1}{4\mu} \int_{\Pi_{ij}} \left[\boldsymbol{\sigma} : \boldsymbol{\sigma} + 2\boldsymbol{\sigma} : \mathbf{T} + \mathbf{T} : \mathbf{T} - \frac{(\text{tr} \boldsymbol{\sigma})^2}{2} - \text{tr} \boldsymbol{\sigma} \text{tr} \mathbf{T} - \frac{(\text{tr} \mathbf{T})^2}{2} \right] d\mathbf{x} \\ &= \mathbf{w}(\boldsymbol{\sigma}) + 2w(\boldsymbol{\sigma}, \mathbf{T}) + \mathbf{w}(\mathbf{T}),\end{aligned}$$

where

$$w(\boldsymbol{\sigma}, \mathbf{T}) = -\frac{1}{4\mu} \int_{\Pi_{ij}} \left[(\boldsymbol{\sigma} : \mathbf{T}) - \frac{\text{tr} \boldsymbol{\sigma} \text{tr} \mathbf{T}}{2} \right] d\mathbf{x}, \quad (\text{C.6})$$

and the corresponding quadratic form:

$$\mathbf{w}(\boldsymbol{\sigma}) = -\frac{1}{4\mu} \int_{\Pi_{ij}} \left[(\boldsymbol{\sigma} : \boldsymbol{\sigma}) - \frac{(\text{tr} \boldsymbol{\sigma})^2}{2} \right] d\mathbf{x}.$$

Hence, $W_3^*(\boldsymbol{\sigma} + \mathbf{T})$ can be written as:

$$W_3^*(\boldsymbol{\sigma} + \mathbf{T}) = \mathcal{L}(\boldsymbol{\sigma}) + \mathcal{L}(\mathbf{T}) + \mathcal{M}(\boldsymbol{\sigma}) + \mathcal{M}(\mathbf{T}) + \mathbf{w}(\boldsymbol{\sigma}) + 2w(\boldsymbol{\sigma}, \mathbf{T}) + \mathbf{w}(\mathbf{T}),$$

where $W_3^*(\boldsymbol{\sigma}) = \mathcal{L}(\boldsymbol{\sigma}) + \mathcal{M}(\boldsymbol{\sigma}) + \mathbf{w}(\boldsymbol{\sigma})$ in these notations.

Taking into account that

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{u}_{per}) = \begin{pmatrix} 2\mu \frac{\partial u_1^{per}}{\partial x} - p_\sigma & \mu \left(\frac{\partial u_1^{per}}{\partial y} + \frac{\partial u_2^{per}}{\partial x} \right) \\ \mu \left(\frac{\partial u_1^{per}}{\partial y} + \frac{\partial u_2^{per}}{\partial x} \right) & 2\mu \frac{\partial u_2^{per}}{\partial y} - p_\sigma \end{pmatrix},$$

and

$$p_\sigma = -\frac{\text{tr} \boldsymbol{\sigma}}{2},$$

we consider

$$\begin{aligned}
2w(\boldsymbol{\sigma}, \mathbf{T}) &= -\frac{1}{2\mu} \int_{\Pi_{ij}} \left[(\boldsymbol{\sigma} : \mathbf{T}) - \frac{\text{tr } \boldsymbol{\sigma} \text{tr } \mathbf{T}}{2} \right] d\mathbf{x} \\
&= -\frac{1}{2\mu} \int_{\Pi_{ij}} \left[\left(2\mu \frac{\partial u_1^{per}}{\partial x} - p_\sigma \right) T_{11} + 2\mu \left(\frac{\partial u_1^{per}}{\partial y} + \frac{\partial u_2^{per}}{\partial x} \right) T_{12} \right. \\
&\quad \left. + \left(2\mu \frac{\partial u_2^{per}}{\partial y} - p_\sigma \right) T_{22} + p_\sigma (T_{11} + T_{22}) \right] d\mathbf{x} \\
&= -\frac{1}{2\mu} \int_{\Pi_{ij}} \left[2\mu \frac{\partial u_1^{per}}{\partial x} T_{11} + 2\mu \frac{\partial u_1^{per}}{\partial y} T_{12} + 2\mu \frac{\partial u_2^{per}}{\partial x} T_{12} + 2\mu \frac{\partial u_2^{per}}{\partial y} T_{22} \right] d\mathbf{x} \\
&= -\int_{\Pi_{ij}} \left[\frac{\partial u_1^{per}}{\partial x} T_{11} + \frac{\partial u_1^{per}}{\partial y} T_{12} + \frac{\partial u_2^{per}}{\partial x} T_{12} + \frac{\partial u_2^{per}}{\partial y} T_{22} \right] d\mathbf{x} \\
&= \int_{\Pi_{ij}} \left[u_1^{per} \frac{\partial T_{11}}{\partial x} + u_1^{per} \frac{\partial T_{12}}{\partial y} + u_2^{per} \frac{\partial T_{12}}{\partial x} + u_2^{per} \frac{\partial T_{22}}{\partial y} \right] d\mathbf{x} \\
&\quad - \int_{\partial \Pi_{ij}} (u_1^{per} n_1 T_{11} + u_1^{per} n_2 T_{12} + u_2^{per} n_1 T_{12} + u_2^{per} n_2 T_{22}) ds \\
&= -\int_{\partial \Pi_{ij}^+} \mathbf{u}_{per} \cdot \mathbf{T} \mathbf{n} ds - \int_{\partial \Pi_{ij}^-} \mathbf{u}_{per} \cdot \mathbf{T} \mathbf{n} ds \\
&\quad - R(\omega^i - \omega^j) \left(\int_{\partial B^i} \mathbf{G}_3^i \cdot \mathbf{T} \mathbf{n} ds + \int_{\partial B^j} \mathbf{G}_3^j \cdot \mathbf{T} \mathbf{n} ds \right), \tag{C.7}
\end{aligned}$$

where the volume integral over Π_{ij} disappears due to divergence-free property of the tensor \mathbf{T} . Continuing (C.7) and taking into account $\mathbf{T} \mathbf{n} = \chi_T^\pm \mathbf{n}$ on $\partial \Pi_{ij}^\pm$ and the fact that \mathbf{u}_{per} satisfies $\frac{1}{R} \int_{\partial \Pi_{ij}^\pm} \mathbf{u}_{per} \cdot \mathbf{n} ds = \pm \beta_{ij}^*$ (this is from the divergence free condition of \mathbf{u}_{per} in Π_{ij}^\pm) we obtain:

$$\begin{aligned}
2w(\boldsymbol{\sigma}, \mathbf{T}) &= -\chi_T^+ \int_{\partial \Pi_{ij}^+} \mathbf{u}_{per} \cdot \mathbf{n} ds - \chi_T^- \int_{\partial \Pi_{ij}^-} \mathbf{u}_{per} \cdot \mathbf{n} ds \\
&\quad - R(\omega^i - \omega^j) \left(\int_{\partial B^i} \mathbf{G}_3^i \cdot \mathbf{T} \mathbf{n} ds + \int_{\partial B^j} \mathbf{G}_3^j \cdot \mathbf{T} \mathbf{n} ds \right) \\
&= -\chi_T^+ R \beta_{ij}^* + \chi_T^- R \beta_{ij}^* - \mathcal{M}(\mathbf{T}) = -\mathcal{L}(\mathbf{T}) - \mathcal{M}(\mathbf{T}).
\end{aligned}$$

Finally, we observe that

$$\mathbf{w}(\mathbf{T}) = -\frac{1}{4\mu} \int_{\Pi_{ij}} \left[(\mathbf{T} : \mathbf{T}) - \frac{(\text{tr } \mathbf{T})^2}{2} \right] d\mathbf{x} = -\frac{1}{4\mu} \int_{\Pi_{ij}} \left[\frac{1}{2} (T_{11} - T_{22})^2 + 2T_{12}^2 \right] d\mathbf{x} \leq 0.$$

Therefore, from all the above we conclude

$$\begin{aligned}
W_3^*(\boldsymbol{\sigma} + \mathbf{T}) &= \mathcal{L}(\boldsymbol{\sigma}) + \mathcal{L}(\mathbf{T}) + \mathbf{w}(\boldsymbol{\sigma}) + 2w(\boldsymbol{\sigma}, \mathbf{T}) + \mathbf{w}(\mathbf{T}) \\
&= W_3^*(\boldsymbol{\sigma}) + \mathcal{L}(\mathbf{T}) - \mathcal{M}(\mathbf{T}) + \mathcal{M}(\mathbf{T}) - \mathcal{L}(\mathbf{T}) + \mathbf{w}(\mathbf{T}) \\
&= W_3^*(\boldsymbol{\sigma}) + \mathbf{w}(\mathbf{T}) \leq W_3^*(\boldsymbol{\sigma}),
\end{aligned}$$

and this inequality holds for any $\mathbf{T} \in F_{per}$. Thus, $\boldsymbol{\sigma}(\mathbf{u}_{per})$ is a maximizer of the functional (6.14).

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